This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

TX 512.1 .W583
White, Emerson E.
School algebra : designed for use in hig
Stanford University Libraries
3 6105 04936 4081

OF MATHEMATICS

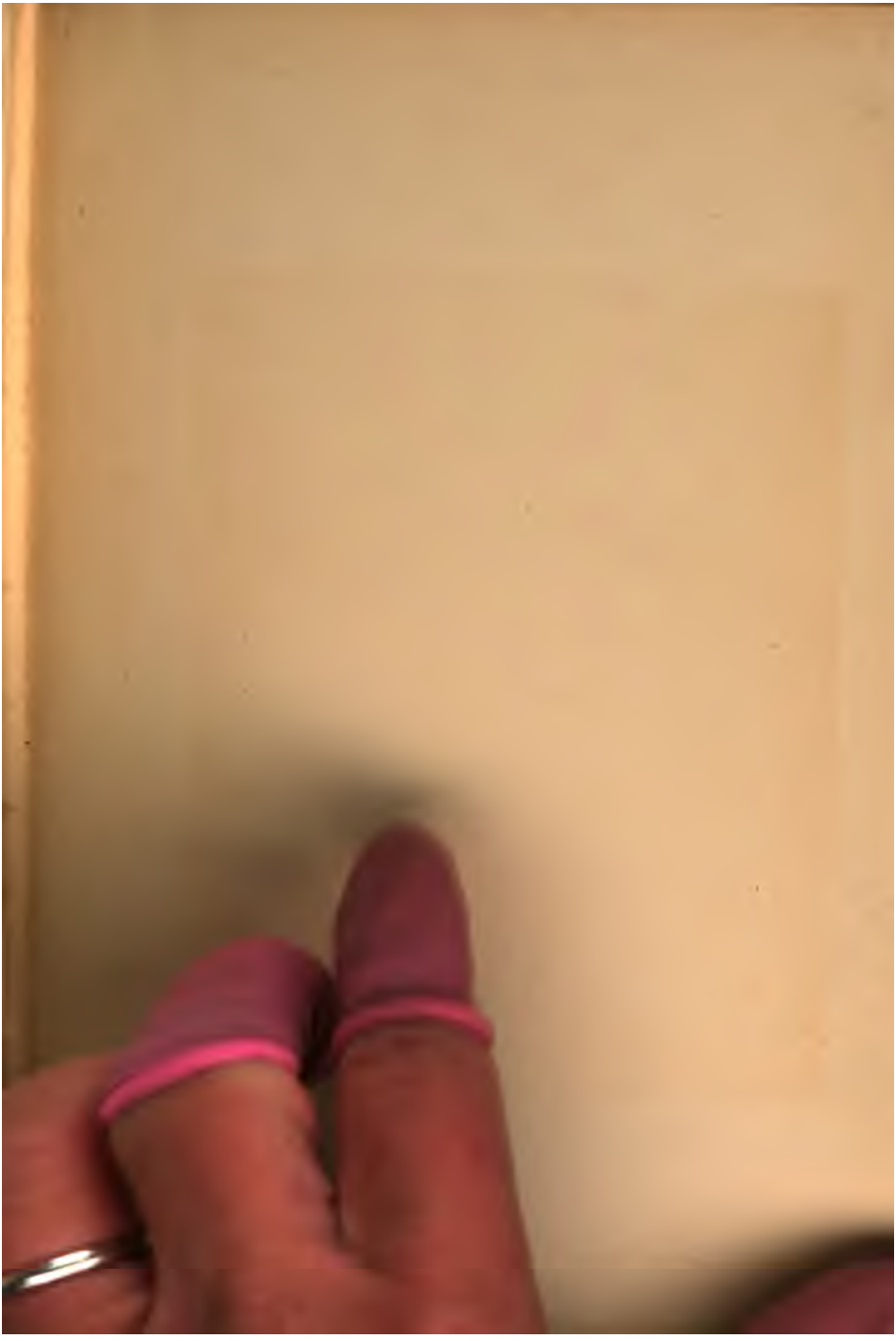
SCHOOL ALGEBRA

E. E. WHITE

SCHOOL OF EDUCATION
LIBRARY

TEXTBOOK
COLLECTION

STANFORD UNIVERSITY
LIBRARIES



WHITE'S SERIES OF MATHEMATICS

A

SCHOOL ALGEBRA

DESIGNED FOR USE IN

HIGH SCHOOLS AND ACADEMIES

BY

EMERSON E. WHITE, A.M., LL.D.

AUTHOR OF "SERIES OF MATHEMATICS," "ELEMENTS OF PEDAGOGY," "SCHOOL MANAGEMENT," ETC.

NEW YORK ·:· CINCINNATI ·:· CHICAGO

AMERICAN BOOK COMPANY

632282
C

WHITE'S MATHEMATICS

Oral Lessons in Number
(FOR TEACHERS)

First Book of Arithmetic
New Complete Arithmetic
School Algebra
Elements of Geometry

WHITE'S PEDAGOGICS
Elements of Pedagogy
School Management

WHITE'S SCHOOL RECORDS
New School Register
Monthly School Record
Teachers' Class Record

COPYRIGHT, 1896, BY AMERICAN BOOK COMPANY.

E P 7

PREFACE.

THIS work is designed for use in high schools and academies, and it covers sufficient ground to meet fully the entrance requirements of the best colleges and universities. It is specially adapted to the first two years of the usual high-school course.

It has been the author's aim to prepare a school algebra that is pedagogically sound, as well as mathematically accurate, and thoroughly adequate for its place and purpose. He has kept in mind the fact that the great majority of the pupils who begin the study of algebra are between thirteen and fifteen years of age, and hence are too young and immature to master successfully a text-book designed for older and more advanced students. These young pupils have as a class a fair knowledge of the analytic and inductive processes of arithmetic; and true pedagogical principles require that this prior training be made as helpful as possible in their introduction to algebra.

The manner in which these ends have been attained is partially indicated by the following statements: —

1. *The early introduction and practical use of the equation.* The equation, properly called "the instrument of algebra," is used from the beginning in the solution of simple problems, thus awakening at the first the pupil's interest in the new study. The pupil has been using the informal equation throughout his entire course in arithmetic, and so finds no difficulty in using the equation in solving problems *in advance of its formal treatment.* The equation is also freely used in proofs and explanations, and in establishing data for inductions and other generalizations. This use of the equation avoids those long and abstruse verbal statements which so baffle the young learner.

2. *The use of arithmetical approaches to algebraic processes and principles,* especially in the first half of the work. Since arithmetic deals with particular numbers, and algebra with both particular and general numbers, the processes and principles of arithmetic afford a natural and easy approach to those of algebra. This conforms to two fundamental principles of procedure in elementary teaching; to wit, "*from the particular to the general,*" and "*from the known to the related unknown.*" These pedagogical principles apply to the teaching of the *elements* of all science, and they are specially helpful in the pupil's introduction to algebra.

3. *The intelligent use of the inductive method.* The introduction of the inductive method is a characteristic feature of modern arithmetic, and its increasing use in algebra establishes a still closer relation between the two studies. In the present work, the inductive method is used, when practica-

ble, and algebraic principles and laws are thus easily and clearly reached. When formal rules are given, they are placed *after* the exercises and problems, as in the author's "Series of Arithmetics," — an arrangement specially commended by the "Committee of Ten." The method of deductive proof is introduced gradually as the pupil advances towards the closing chapters, thus making him somewhat familiar with the method of demonstration which he will subsequently use in geometry.

4. *The immediate application of facts and principles in simple exercises for practice;* this being a marked feature of the chapter on algebraic notation, often so difficult for beginners. Practice exercises not only help fix the facts and principles taught in the memory, but they greatly increase the clearness with which they are comprehended by the young learner. A principle that may not be fully understood from its statement is often clearly grasped in its application.

Attention is specially called to *the character of the exercises and problems.* The essential result to be attained by the pupil in the study of the elements of algebra is FACILITY AND ACCURACY IN ALGEBRAIC PROCESSES. This is necessary to all satisfactory future progress, and, to this end, there must be abundant and appropriate practice. Great pains have been taken to make the exercises and problems in this work adequate in *number*, *variety*, and *grading*. There has been a careful exclusion of problems believed to be too difficult. Such problems not only discourage the learner, but call for an unprofitable use of time and energy.

Progressive teachers will be pleased to find a number of subjects and processes not given in the school algebras in general use. There will also be found throughout the work new and elegant solutions, and other new features of special interest and value. It is believed that few text-books have been prepared with greater care, or with more earnest effort to ascertain and meet the needs of the schools. *The result is a progressive modern algebra.*

The author gratefully acknowledges his indebtedness to Professor JOHN MACNIE, of the University of North Dakota, for many exercises and problems, and for other contributions of subject-matter; to Professor M. C. STEVENS, in charge of the department of mathematics in Purdue University, Indiana, for a critical reading of the manuscript, and for many helpful suggestions, including new solutions and proofs, and other material; to Professor E. A. LYMAN, of the University of Michigan, for the critical reading of the manuscript, valuable suggestions, and other help; and also to several experienced teachers of algebra in high schools, who have rendered important assistance.

COLUMBUS, O.

CONTENTS.

CHAPTER		PAGE
I.	INTRODUCTION	7
	Algebraic Equations	7
II.	ALGEBRAIC NOTATION	16
	Positive and Negative Numbers	30
	Laws	33
	Equations and Problems	39
III.	ADDITION AND SUBTRACTION	43
IV.	MULTIPLICATION AND DIVISION	54
	Fractional Coefficients	67
	Detached Coefficients	68
	Synthetic Division	71
V.	SIMPLE EQUATIONS	74
VI.	FORMULAS	85
	Special Forms in Multiplication	85
	Division by Binomial Factors	90
VII.	FACTORING	95
	Special Methods of Factoring	95
	General Method of Factoring Trinomials	108
	Factoring by Synthetic Division	113
VIII.	COMMON FACTORS AND MULTIPLES	119
IX.	FRACTIONS	130
	Reduction of Fractions	132
	Addition and Subtraction of Fractions	139
	Multiplication and Division of Fractions	143
X.	SIMPLE EQUATIONS CONTAINING FRACTIONS	152
XI.	SIMULTANEOUS EQUATIONS	169
	Simple Equations with Two Unknown Numbers	169
	Equations with Three or More Unknown Numbers	179
XII.	INVOLUTION AND EVOLUTION	186
	Powers	186
	Roots	193
XIII.	RADICALS	209
	Reduction of Radicals	210
	Addition and Subtraction of Radicals	215
	Multiplication of Radicals	216
	Division of Radicals	219

CHAPTER		PAGE
XIII.	RADICALS (*continued*).	
	Involution and Evolution of Radicals	222
	Equations involving Radicals	226
	Imaginary Numbers	228
XIV.	FRACTIONAL AND NEGATIVE EXPONENTS	234
XV.	QUADRATIC EQUATIONS	239
	Incomplete Quadratics	240
	Complete Quadratics	243
	Literal Quadratics	253
	Equations Quadratic in Form	257
XVI.	SIMULTANEOUS QUADRATIC EQUATIONS	265
	General Methods of Solution	265
	Special Methods	269
XVII.	INEQUALITIES	273
XVIII.	RATIO, PROPORTION, VARIATION	277
	Ratio	277
	Proportion	281
	Variation	287
XIX.	PROGRESSIONS	293
	Arithmetical Progression	293
	Geometrical Progression	300
	Harmonic Progression	307
XX.	LOGARITHMS	309
	Principles	311
	Table with Tabular Differences	319
	Applications to Numerical Processes	322
XXI.	UNDETERMINED COEFFICIENTS AND APPLICATIONS	329
	Resolution of Fractions	330
	Expansion of Fractions into Series	332
	Binomial Formula	334
XXII.	DETERMINANTS	340
	Determinants of the Third Order	343
	Determinants of Any Order	347
	Properties	350
XXIII.	CURVE TRACING	355
	Geometrical Representation of Equations	357
	Geometrical Representation of the Roots of an Equation,	362
XXIV.	PERMUTATIONS AND COMBINATIONS	364
	Permutations	364
	Combinations	367
APPENDIX		369
ANSWERS		375

ALGEBRA.

CHAPTER I.

INTRODUCTION.

ALGEBRAIC EQUATIONS.

1. If a denotes a certain number, $3a$ will denote 3 times the number; $4a$, 4 times the number; and so on.

1. If a denotes a number, what will denote 5 times the number? Seven times the number?

2. If n denotes a number, what will $5n$ denote? $8n$? $12n$?

3. If x denotes the number of feet in a rod, what will denote the number of feet in 3 rods? In 6 rods? In 12 rods?

4. If x denotes the number of bushels of apples in a barrel, what will denote the number of bushels in 5 barrels? In 15 barrels? In 20 barrels?

5. If x denotes a man's age, what will denote 3 times his age? 8 times his age? $\frac{2}{3}$ of his age?

2. In this introductory chapter, the signs $+$, $-$, $\times$, $\div$, and $=$ have the same meaning and use as in arithmetic.

The expression $7 + 5$ denotes that 5 is to be added to 7, and $7 - 5$ denotes that 5 is to be subtracted from 7. In like manner, $a + b$ denotes that the number represented by b is to be added to the number represented by a, and $a - b$ denotes that the

number represented by b is to be subtracted from the number represented by a.

6. The sum of $3a$ and $2a$ is expressed by $3a + 2a$, which is $5a$. In like manner express the sum of $5a$ and $4a$. How many times a in the sum?

7. Express by the sign + the sum of $4x$ and $6x$. How many times x in the sum?

8. Express the sum of $3x$, $2x$, and $5x$. How many times x in the sum?

9. If x denotes A's age, and $3x$ B's age, what will express the sum of their ages? How many times x in the sum?

10. If x denotes the cost of a chair, $3x$ the cost of a table, and $5x$ the cost of a lounge, what will express the cost of the three articles? How many times x in the cost?

11. An orchard contains 5 rows of pear trees, 4 rows of peach trees, and 7 rows of apple trees. If x denotes the number of trees in each row, what will express the number of trees in the orchard? How many times x in the number of trees?

12. A number is divided into two parts such that the greater part is 4 times the less. If x denotes the less part, what will denote the greater? What will express the sum of the two parts? How many times x in the number?

13. The difference expressed by $5a - 3a$ is $2a$. What is the difference expressed by $7a - 4a$? How many times a in the difference?

14. What is the difference expressed by $9a - 5a$? $13a - 7a$? $20a - 15a$? $17a - 9a$? $21a - 12a$?

15. What is the difference expressed by $7x - 4x$? $8x - 3x$? $12x - 7x$? $15x - 6x$? $21x - 7x$?

16. If $5x$ denotes A's age, and $3x$ denotes B's age, what will express the difference of their ages? How many times x years is A older than B?

3. If $4x = 20$, $1x$, or simply x, is $\frac{1}{4}$ of 20, which is 5; and if x equals 5, $3x$ is three times 5, which is 15. The number 5, which x here denotes, is called the **value** of x.

17. If $3x = 15$, what is the value of x? Of $5x$? Of $9x$?

18. If $3x + 4x = 35$, what is the value of x? Of $4x$?

19. If $6x + 4x = 90$, what is the value of x?

20. If $7x + 5x + 3x = 45$, what is the value of x?

21. If $12x - 7x = 30$, what is the value of x?

22. If $7x + 4x - 5x = 24$, what is the value of x?

23. If $8x + 2x - 7x = 21$, what is the value of x?

4. The expression $7 + 5 = 3 \times 4$ denotes that the sum of 7 and 5 is equal to the product of 3 and 4. In like manner, $a + b = c \times d$ denotes that the sum of the numbers represented by a and b is equal to the product of the numbers represented by c and d.

5. The equality of two numbers may be expressed by the sign $=$, which is read "equals" or "is equal to." Thus, $2x + 3x = 25$ is read "two x plus three x equals 25."

6. An expression denoting the equality of two numbers is called an **equation.** Thus, $2x + 3x = 25$ is an equation.

7. An equation in which all of the numbers are expressed by figures is called an **arithmetical equation.** An equation in which one or more of the numbers is expressed by letters is called an **algebraic equation.** Thus, $8 \times 5 - 16 = 6 \times 4$ is an arithmetical equation; and $2x + 5 = 15$, $a + b = 20$, and $a - b = c$, are algebraic equations.

8. The solution of problems by means of an algebraic equation is called the **algebraic method.**

The first step in the solution of a problem by the algebraic method is to state the conditions of the problem in the form

of an equation, and the second step is to find the value of the unknown number.

The finding of the value of the unknown number in an equation is called the **solution** of the equation.

9. The advantage of the algebraic method of solving problems is best shown by its actual use in the solution of problems which can also be readily solved by the methods of arithmetic.

Take, for example, this problem:

A's age is twice B's age, and the sum of their ages is 60 years. What is the age of each?

ARITHMETICAL SOLUTION.

By the conditions of the problem, A's age is twice B's age, and B's age plus twice B's age, or 3 times B's age, is 60 years. Hence B's age is one third of 60 years, which is 20 years; and A's age is twice 20 years, which is 40 years.

ALGEBRAIC SOLUTION.

Let x denote B's age; then $2x$ will denote A's age, and, by the conditions of the problem, we have

$$x + 2x = 60;$$

whence $3x = 60;$

and $x = 20$, B's age;

and $2x = 40$, A's age.

Hence B's age is 20 years, and A's age is 40 years.

10. In the algebraic statement of a problem, each number is considered as *abstract*. Thus, since 20 years $= 20 \times 1$ year, x in the above solution represents 20, the concrete unit (1 year) being omitted. To express the number of years, the abstract value of x, when found, is considered as multiplied by the omitted concrete unit (1 year).

Problems.

1. A and B together have $45, and A has twice as much money as B. How much money has each?

Arithmetical Solution.

Twice B's money = A's money;
then twice B's money + B's money = $45.
Hence 3 times B's money = $45;
whence B's money = ⅓ of $45 = $15,
and A's money = 2 × $15 = $30.

Algebraic Solution.

Let x = B's money;
then $2x$ = A's money,
and $x + 2x = 3x$ = B's and A's money.
Hence $3x = 45$;
whence $x = \frac{1}{3}$ of $45 = 15$,
and $2x = 2 \times 15 = 30$.
Hence A has $30, and B $15.

Solve the next eight problems first arithmetically and then algebraically.

2. A father and his son earn together $56 a month, and the father earns 3 times as much as the son. How much does each earn?

3. A man paid $24 for a coat and vest, and the coat cost 5 times as much as the vest. What was the cost of each?

4. The sum of two numbers is 42, and the greater number is 5 times the less. Find the numbers.

5. Divide $36 into two parts such that the greater shall be 3 times the less.

6. Cut a piece of tape 30 yards long into two pieces such that the longer piece shall contain 5 times as many yards as the shorter.

7. A and B together own 150 sheep, and A owns twice as many sheep as B. How many sheep does each own?

8. The sum of two numbers is 90, and 4 times the less number equals the greater. What are the numbers?

9. If a number be increased by twice itself, the result will be 60. What is the number?

10. The difference between two numbers is 24, and the greater number is 4 times the less. What are the numbers?

Let x = the less number;
then $4x$ = the greater number,
and $4x - x = 3x$ = their difference.
Hence $3x = 24$;
whence $x = 8$, the less number,
and $4x = 32$, the greater number.

11. A father's age is 3 times the age of his son, and the difference of their ages is 30 years. What is the age of each?

12. Divide a number into two parts such that the greater part will be 4 times the less, and their difference 81. What is each part? What is the number?

13. Three times the cost of a saddle was the cost of a harness, and the harness cost \$ 12 more than the saddle. What was the cost of each?

14. A school enrolls 180 pupils, and there are 20 more girls than boys. How many pupils of each sex in the school?

Let x = number of boys;
then $x + 20$ = number of girls,
and $2x + 20$ = number of pupils.
Hence $2x + 20 = 180$.
Subtracting 20, $2x = 160$;
whence $x = 80$, number of boys;
$x + 20 = 100$, number of girls.

If a number $+ 5 = 35$, then the number $= 35 - 5$, which is 30. In like manner if $x + 5 = 35$, then $x = 35 - 5 = 30$; and if $x - 5 = 25$, then $x = 25 + 5 = 30$. It is thus seen that a number may be added to or subtracted from both members of an equation without affecting their equality.

15. A pole 120 feet long fell and broke into two pieces, one piece being 30 feet longer than the other. What was the length of each piece?

16. The sum of two numbers is 120, and their difference is 20. What are the numbers?

17. Divide $1800 between two persons, giving to one $650 more than to the other.

18. A man bought a watch and chain for $85, and the cost of the watch was $5 more than 3 times the cost of the chain. What was the cost of each?

19. In a certain election 364 votes were polled by the two parties, and one party had 48 majority. How many votes were cast by each party?

20. In a certain village, containing 327 persons, there are 15 more women than men, and twice as many children as there are men and women together. How many of each in the village?

21. Three men, A, B, and C, bought a mill for $12,000. A paid twice as much as B, and C paid 5 times as much as B. How much did each pay?

22. Divide a piece of cloth containing 42 yards into three pieces, making the second piece 3 times the length of the first, and the third piece one half of the length of the other two pieces together.

23. Cut a cord 45 feet long into two pieces such that one piece shall be 15 feet longer than the other.

24. Divide $8400 among A, B, and C, giving to B twice as much as to A, and to C twice as much as to B.

25. Divide $18 among A, B, and C, giving to A twice as much as to B, and to C twice as much as to A and B together.

26. Divide $1800 among three persons, giving to the second $200 more than to the first, and to the third $200 more than to the second.

27. A is twice as old as B, and B is 15 years younger than C, and the sum of their ages is 95 years. How old is each?

28. Four times a certain number is 45 more than the number. What is the number?

29. Divide $140 between two men, giving one $20 more than the other.

30. The sum of A's and B's ages is 70 years, and A's age is 4 times B's age. What is the age of each?

31. A school enrolls 240 pupils, and twice the number of boys equals the number of girls. How many of each are enrolled?

32. A man sold a horse and a buggy for $180, and received twice as much for the horse as for the buggy. What was the price of each?

33. A mother is 3 times as old as her daughter, and the difference of their ages is 30 years. How old is each?

34. Divide $125 between A and B so that A shall receive $45 more than B.

35. The sum of two numbers is 75, and their difference is 15. What are the numbers?

36. A man owns two farms which together contain 200 acres, and the larger farm contains 42 acres more than the smaller. How many acres in each farm?

37. A farmer who owned a flock of sheep bought 3 times as many sheep as he had, and then had 248 sheep. How many sheep did he buy?

38. A tree 90 feet long was broken by the wind, and the part left standing was 20 feet shorter than the part broken off. What was the length of each part?

39. A and B are partners in business, and A's capital is $500 less than twice B's, and their total capital is $5500. How much capital has each?

40. An estate of $22,050 was bequeathed to a widow and two sons. The sons received equal shares, and the widow twice as much as the two sons together. How much did each receive?

41. Divide a line 64 inches long into two parts such that the longer shall be 8 inches less than twice the shorter.

42. A banker paid $102 in ten-dollar, five-dollar, and two-dollar bills, using the same number of bills of each kind. How many bills of each kind did he use? How many bills in all?

43. A school, enrolling 76 pupils, is divided into three classes. There are twice as many pupils in the second class as in the first, and 16 more pupils in the third class than in the second. How many pupils in each class?

44. A father is twice as old as his son, and the sum of their ages less 12 years is 60 years. How old is each?

45. A mother is 3 times as old as her daughter less 10 years, and the sum of their ages is 50 years. How old is each?

46. A's age is twice B's, and B's age is twice C's, and the sum of all their ages is 126 years. What is the age of each?

47. A father and his two sons earn $140 a month, and the father earns twice as much as the elder son, and the elder son twice as much as the younger. How much does each earn?

48. At an election there were three candidates. The first received 40 votes more than the second, and 65 votes more than the third, and the whole number of votes cast was 306. How many votes did each candidate receive?

CHAPTER II.

ALGEBRAIC NOTATION.

SYMBOLS REPRESENTING NUMBERS.

11. In arithmetic, numbers are represented by words and by the Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, called **figures**; and the numbers thus represented are *definite known numbers*. Thus, the word *eight* and the figure 8 alike represent a definite number of ones or units.

The figure 0, called **zero**, denotes the absence of number.

In algebra, numbers are represented by figures and by letters. Figures in algebra, as in arithmetic, represent definite known numbers, while letters represent any numbers, whether known or unknown.

Figures are called *arithmetical* symbols of numbers; and letters, general or *algebraic* symbols.

12. Known numbers, when expressed by letters, are usually represented by the *first* letters of the alphabet; as a, b, c, etc.

13. Unknown numbers, that is, numbers whose values are not given or determined, are usually represented by the *final* letters of the alphabet; as x, y, z.

This distinction in the use of the first and the final letters of the alphabet is not always observed. Any letter may be used to represent an unknown number; and numbers may be represented by the letters of any alphabet.

In the Roman notation, numbers are represented by the letters *I*, *V*, *X*, *L*, *C*, *D*, *M*, which denote definite known numbers, the same as figures.

14. The term *quantity* is used in algebra as synonymous with *number*. A number expressed by letters, or by letters and figures in combination, may be called an algebraic quantity or an algebraic number. In this treatise the term *number* instead of *quantity*, is generally used.

SIGNS OF OPERATION.

15. In both arithmetic and algebra the operations to be performed with numbers are indicated by characters, called **signs.**

16. The sign $+$, called **plus,** indicates that the number after the sign is to be added to the number before it. Thus, $7 + 5$, read "7 plus 5," denotes that the number 5 is to be added to the number 7; and $a + b$ denotes that the number represented by b is to be added to the number represented by a, or, more briefly, that the number b is to be added to the number a.

17. The sign $-$, called **minus,** indicates that the number after the sign is to be subtracted from the number before it. Thus, $7 - 5$, read "7 minus 5," denotes that the number 5 is to be subtracted from the number 7; and $a - b$ denotes that the number represented by b is to be subtracted from the number represented by a, or, more briefly, that the number b is to be subtracted from the number a.

The addition of a and b is denoted by $a + b$, and the subtraction of b from a by $a - b$.

1. Express the addition of x and y; $2x$ and $3y$; 2 and y.

2. Express the addition of a and $2b$; $2a$ and $3b$; a, $2b$, and 15; $3a$, $2b$, and $5c$; 5, $4a$, and $6b$.

3. Express the subtraction of y from x; x from $3y$; 2 from x; x from 2; $3y$ from $5x$.

18. The sign $\times$, called the **sign of multiplication,** and read "multiplied by" or "times," indicates that the two numbers

between which it is placed are to be multiplied together. Thus, $a \times b$ denotes that the number a is to be multiplied by the number b, or the number b by the number a. The numbers multiplied together are called **factors**; and the result obtained by multiplying together the factors, the **product.**

19. When one or both of the numbers to be multiplied are represented by letters, their multiplication may be expressed (1) by the sign $\times$ between them, as $a \times b$ or $3 \times b$; (2) by a dot between the numbers, as $a \cdot b$ or $3 \cdot b$; or (3), as is more common, by writing the numbers together, as ab or $3b$.

When numbers are expressed by figures only, their multiplication is indicated by the sign $\times$ or the dot, as 12×5 or $12 \cdot 5$.

20. The sign $\div$, called the **sign of division,** and read "divided by," indicates that the number before it is to be divided by the number after it. Thus, $a \div b$ denotes that the number a is to be divided by the number b.

The division of one number by another may also be indicated by writing the dividend over the divisor in the form of a fraction. Thus, the division of a by b may be indicated by $a \div b$, or $\frac{a}{b}$, often written a/b.

21. In arithmetic the operations indicated by signs can actually be performed, but in algebra many operations can only be indicated. Hence in algebra the signs $+$, $-$, $\times$, and $\div$ each may indicate both an operation and its result. Thus, $a + b$ denotes the addition of a and b, and also their sum; $a - b$, the subtraction of b from a, and also their difference; $a \times b$ or $a \cdot b$, the multiplication of a and b, and also their product; and $a \div b$ or $\frac{a}{b}$, the division of a by b, and also their quotient.

The expression ab indicates the product of a and b, the sign $\times$ or $\cdot$ being omitted; and usually $\frac{a}{b}$ indicates simply the

quotient of a divided by b. In arithmetic, $\frac{3}{5}$ denotes the quotient of 3 by 5 as well as a fraction of a unit.

4. Express the addition of x and y; of x, y, and z.

5. Express the addition of a, $2b$, and $3c$; of r, m, and n.

6. Express the subtraction of n from m; of y from x.

7. Express the subtraction of ab from xy; of 2 from ax.

8. Express the multiplication of x by y in three ways.

9. Express the multiplication of x, y, and z in three ways.

10. Write without $\times$ or $\cdot$ the product of a and x; of a, x, y; of $5, a, b, c$; of $3, a, x, y$; of $10, x, y, z$.

11. Write without $\times$ or $\cdot$ the product of $3, ab, c$; of $12, x, yz$.

12. Express the division of x by y in two ways.

13. Express the division of an by $3xy$ in two ways.

14. Let b denote the base, and r the rate per cent, and then express the percentage.

SUGGESTION. The rate per cent is the number of hundredths taken, and hence the percentage is expressed by $b \times r$, or br.

15. Let p denote the percentage, and r the rate per cent, and express the base.

16. Let p denote the principal, r the rate per cent, and t the time in years, and express the interest.

17. Let p denote the principal, and i the interest, and express the amount.

22. The **factors** of a number are the numbers which, multiplied together, will produce it. Thus, 5 and 7 are the factors of 35; a and b, the factors of ab; and 3, x, y, the factors of $3xy$.

A number that contains no integral factor except itself and 1 is called a prime number (§ 176).

If one factor of a product is equal to 0, the product is equal to 0, whatever may be the finite values of the other factors.

23. Factors expressed by figures are called **numerical factors**, and factors expressed by letters are called **literal factors**. Thus, 3 is a numerical factor of $3ab$, and a and b literal factors.

24. A known factor prefixed to another factor or set of factors is called a **coefficient**. Thus, in $5a$, 5 is the coefficient of a; and in $3ax$, 3 is the coefficient of ax. a may be considered the coefficient of x in ax, and $3a$ in $3ax$.

25. When a coefficient is expressed by figures, it is called a **numerical coefficient**; and when it contains one or more letters, it is called a **literal coefficient**. Thus, the a in ax, and the $3a$ in $3ax$, are the literal coefficients of x.

When an algebraic number has no numerical coefficient expressed, the coefficient 1 is understood. Thus, a is the same as $1a$, and xy the same as $1xy$.

26. The term *coefficient* means *co-factor*, and, in this sense of the term, any factor or set of factors of a product are coefficients of the other factors. Thus, in $3ax$, 3 is the coefficient of ax, $3a$ the coefficient of x, a the coefficient of $3x$, and x the coefficient of $3a$.

The term *coefficient* is, however, usually applied to the known factors, numerical or literal, which are placed before the other factors of a product, showing how many times they are taken.

27. A **power** of a number is the product obtained by taking the number one or more times as a factor. Thus, 25 (5×5) is the second power, or square, of 5; 27 $(3 \times 3 \times 3)$ is the third power, or cube, of 3; and $a \times a \times a \times a$ is the fourth power of a.

28. The power of a number may be indicated by writing at the right of the number, and a little above it, a number to denote how many times the given number is taken as a *factor*. Thus, the third power of 5 is denoted by 5^3; the fourth power of a, by a^4; and the nth power of a, by a^n.

29. The number that is taken one or more times as a factor is called the **base** of the power; and the number at its right, denoting how many times it is taken as a factor, is called the **exponent** of the power. Thus, in x^5, x is the base, and 5 the exponent.

30. When an exponent is a figure, as in x^5, it is called a **numerical exponent**; and when it is a letter, as in x^n, it is called a **literal exponent.**

The first power of a number is the number itself; and hence, when a number has no exponent expressed, the exponent 1 is understood. Thus, $6 = 6^1$, and $x = x^1$.

31. The exponent of a number which is expressed by two or more orders is written at the right of the right-hand figure. Thus, 25^3 denotes the third power of 25; and 3.04^4, the fourth power of 3.04. $(\frac{3}{4})^2$ denotes the square of $\frac{3}{4}$.

32. The exponent of a literal number denotes the power of the letter only to which it is attached. Thus, a^2b^3 denotes the product of a^2 and b^3, and is read "a square b cube;" and $3a^4x^3$ denotes the product of 3, a^4, and x^3, and is read "$3a$ fourth x cube."

33. The **reciprocal** of a number is unity divided by the number. The reciprocal of 5 is $\frac{1}{5}$; and of a, $\frac{1}{a}$.

18. Write the prime factors (omitting 1) of 10; of 42; of $11ax$; of $6a^2x^3y$; of $5ax^2y^2$; of $21a^2b^2xy^3$.

19. Write the numerical coefficient of $5a^2$; of $3x^2$; of $7x^2y^2$; of ax^2y^3; of $3az^2$; of $\frac{2}{3}axy^2$; of $a^2b^2c^3$.

20. Write the exponent (including 1) of each factor in ab^2x^3; in $5a^3xy^2$; in $6^2ab^3x^5$; in $\frac{2}{3}a^2bc^3$; in $\frac{3}{5}xy^3z^2$.

21. Express by exponents the square of ax; of abx; of $2axy$.

22. Express by exponents the cube of 13; of ab; of $2abx$; of $\frac{2}{3}$.

23. Express the sum of the squares of a, b, and c; the sum of the squares of x, y, and z.

24. Express the sum of the cubes of 4, a, m, and x; the sum of the cubes of a, b, and $2c$.

25. Express the sum of the first three powers of x; the sum of the first five powers of x.

26. Assume the first number to be the minuend, and express the difference of the squares of x and y; of m and n; of 3 and a; of $2a$ and b; of $3x$ and $2y$.

27. Express the product of the cubes of a, b, and c; of the squares of x, y, and z.

28. Express in the form of a fraction the cube of a divided by the square of b; the square of x divided by the cube of ay.

34. The **root** of a number is one of the equal factors which multiplied together will produce the number, or is the number itself. Thus, 5 is the second or square root of 25; 4 is the third or cube root of 64; and a is the fourth root of a^4.

35. The root of a number is indicated by the character $\sqrt{\ }$, called the **radical sign**, with a number placed above it, as $\sqrt[3]{\ }$, called the **index** of the root. Thus, $\sqrt[2]{9}$ denotes the second or square root of 9, which is 3; $\sqrt[3]{125}$, the third or cube root of 125, which is 5; and $\sqrt[4]{a}$, the fourth root of a.

36. When no index is expressed, the sign $\sqrt{\ }$ indicates the square root. Thus $\sqrt{16}$ and $\sqrt[2]{16}$ each denotes the square root of 16. The first root of a number is the number itself, and hence $\sqrt[1]{5} = 5$.

The expression $\sqrt{81}$ is interpreted or read "the square root of 81;" $\sqrt[4]{75+6}$, "the fourth root of the sum of 75 and 6;" $\sqrt[3]{64 \times 8}$, "the cube root of the product of 64 and 8;" and $\sqrt[3]{\frac{8}{27}}$, "the cube root of $\frac{8}{27}$."

29. Copy and read $\sqrt{a-b}$; $\sqrt[3]{a^3x^5}$; $\sqrt[4]{a^2-b^2}$; $\sqrt[1]{214}$.

30. Copy and read $\sqrt{x^3}$; $\sqrt{x^2y^2}$; $\sqrt[3]{x^3y^3}$; $\sqrt[4]{x^2+y^2}$; $\sqrt[n]{x^m}$.

31. What is the value, or indicated root, of $\sqrt{25}$? Of $\sqrt[3]{64}$? Of $\sqrt{144}$? Of $\sqrt[4]{256}$?

32. What is the indicated root of $\sqrt{72-8}$? Of $\sqrt[3]{45-18}$? Of $\sqrt[2]{35+14}$? Of $\sqrt[3]{16\times 4}$?

33. Express the square root of 80; of ab; of a^2-b^2; of x^2+y^2.

34. Express the cube root of 18; of 15×8; of $a+b$; of a^2x^3.

SIGNS OF RELATION.

37. The sign $=$, called the **sign of equality,** denotes that the numbers between which it is placed are equal (§ 5).

38. When the equality of two numbers is expressed by the sign $=$, the numbers are said to be **equated,** and the result is called an **equation.** Thus, by equating x and $a+b$, we form the equation $x=a+b$.

39. The sign $>$ or $<$, called the **sign of inequality,** denotes that the numbers between which it is placed are *unequal,* the opening of the sign or angle being towards the greater number. Thus, $a>b$ denotes that *a is greater than b,* and $a<b$ denotes that *a is less than b.*

The symbol $\neq$ denotes *not equal to*; $\not>$, *not greater than*; $\not<$, *not less than.* They are sometimes used.

40. The sign $\therefore$, called the **sign of deduction,** stands for *hence* or *therefore.*

41. The sign $\cdots$ or ---, called the **sign of continuation,** stands for the words *and so on.* Thus, $1+4+7+10\cdots$ is read "$1+4+7+10$, *and so on.*"

Express the following statements by means of the proper signs:

1. The sum of a and b is equal to c.
2. The sum of x and y is equal to the product of m and n.
3. Four times a is equal to the sum of 3 times c and twice b.

4. A certain number increased by 5 is equal to 3 times the number decreased by 15.

5. The square of a number increased by 5 times the number is equal to 94.

6. The sum of a and b is greater than their difference.

7. The sum of x and y is less than their product.

SIGNS OF AGGREGATION.

42. The parenthesis ( ), the brackets [ ], the braces { }, the vinculum $\overline{\quad}$, and the bar | are called **signs of aggregation.**

43. The first four of these signs all denote that the numbers inclosed are to be treated as one number; that is, are to be taken collectively. Thus, $7 \times (5 - 3)$, $7 \times [5 - 3]$, $7 \times \{5 - 3\}$, and $7 \times \overline{5 - 3}$, all denote that 7 is to be multiplied by the difference of 5 and 3; and each sign of aggregation may be removed if 2 be substituted for $5 - 3$.

44. The vinculum is used in connection with the radical sign, as $\sqrt{28 + 6}$. The line between the numerator and the denominator of a fraction also acts as a vinculum. Thus, $\frac{a+b}{5} = \overline{a + b} \div 5$.

45. The bar denotes that the number at its right and the number at its left are to be multiplied together. Thus,

$x^2 \left| \begin{matrix} 3xy \\ -2y^2 \\ +3y^4 \end{matrix} \right.$ indicates that $3xy - 2y^2 + 3y^4$ is to be multiplied by x^2.

Find the value of

1. $12 + (7 - 3)$.
2. $25 - (13 - 6)$.
3. $23 - (13 + 4)$.
4. $17 + (7 \times \overline{5 - 3})$.
5. $9 \times 4 - (15 + 3)$.
6. $23 + (8 - 3 \times 2)$.
7. $8(7 + 5) - (20 + 4)$.
8. $32 - [(4 \times 6) - 12]$.

9. $9\{6-3\}-3\{4+2\}$.
10. $(5+4)6-\overline{7-4}$.
11. $36-\overline{15-(5\times 2)}$.
12. $8(12-7)-4(5-2)$.

46. When the numbers within a parenthesis, or other sign of aggregation, are represented by letters, the operations indicated cannot always be performed, but the sign of aggregation can be removed in the manner indicated below:

$$(1)\quad a+(b-c)=a+b-c.$$

$$(2)\quad \begin{cases} a-(b+c)=a-b-c. \\ a-(b-c)=a-b+c. \end{cases}$$

$$(3)\quad a\times(b-c)=a\times b-a\times c=ab-ac.$$

$$(4)\quad a\div(b-c)=\frac{a}{b-c}.$$

It is to be specially noted that when the parenthetical expression is preceded by the sign $-$, as in (2) above, the signs $+$ and $-$ within it must be changed ($+$ to $-$, and $-$ to $+$) when the parenthesis is removed.

The reason is obvious. The subtrahend in $a-(b+c)$ is $b+c$, and, if b be first subtracted from a, c must be subtracted from the result. The subtrahend in $a-(b-c)$ is $b-c$, and, if b be first subtracted from a, c must be added to the result, that is, $a-(b-c)=a-b+c$.

Remove the parentheses in each of the following expressions, and reduce the result to its simplest form:

13. $x-(x-y)$.
14. $4x-(2x+y)$.
15. $x+y-(x-y)$.
16. $a^2-(a^2-4)$.
17. $ax-(ax+a)$.
18. $3ab-(2ab-b)$.
19. $a^2b-(a^2b-b+4)$.
20. $a(2+b)+2(a+b)$.
21. $(a+b)-(a-b)$.
22. $5(a+b)-(3a-2b)$.
23. $(a-b)b-(ab+b^2)$.
24. $2(ab-b^2)-(2a-b)b$.
25. $x^2-2xy+y^2-(x^2+y^2)$.
26. $x^2-y^2-(x^2-xy+y^2)$.

53. A polynomial that consists of two terms, as $a + b$, is called a **binomial**; and one that consists of three terms, as $a^2 + 2ab + b^2$, is called a **trinomial.**

An algebraic expression that consists of only one term is also called *simple;* and one that consists of two or more terms, *compound.*

54. **Like terms** are algebraic terms that have the same *literal* factors. Thus, abx, $3abx$, and $-8abx$ are like terms. Like terms are also called **similar terms.**

55. The **degree** of a term is the number of its literal factors. Thus, a and $5x$ are each of the first degree; $3x^2$ and $3ab$, each of the second degree; and $5a^2x^2$, a^2bx, and $-3cx^3$, each of the fourth degree.

56. A polynomial is **homogeneous** when all its terms are of the same degree. Thus, the polynomial $3x^3 - 6x^2y + 5xy^2 + y^3$ is homogeneous, each term being of the third degree.

Copy and read the following algebraic expressions; then give the number of terms in each, and the degree of each term:

1. $3ab - 3a^2 + 6c$.
2. $5ab^2 + 3a^3 - 2ab^3$.
3. $4x^2 - 3xy^2 + 2y^4$.
4. $x^2 + 2xy + y^2$.
5. $15a^2x^2 - 6ax^3 + 5x^4$.
6. $4a^2b - \frac{5a^2b^2}{3} - a^2b^3$.
7. $3ax^2 + 11a^2b^2x + 5a^3y - 9a^4y^2$.
8. $3a^2x + 5(a^2 + x^2) - a^2x^2$.
9. $(x + y)(x - y) - x^2 - xy - y^2$.
10. $3xy - \frac{4xy^2}{5} + \frac{x + 4}{3}$.
11. $3(a + b) - 5(a - b) + 3ab$.
12. $2a - \frac{3a + 6}{4} + 4(a + 5)$.
13. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
14. $x^2 + 2xy + y^2 - 2(x^2 + y^2) + x^2y^2$.
15. $7a^2c^3 + 3a^4c^2 - 5a^6c - \frac{a + b}{c}$.
16. $3a^2x - \frac{3ax^2}{2} - \frac{3x^3 - a^2x}{4} - 3(a^2 - x^2)$.
17. $x^3y + 3x^2y^2 + 3xy^3 + y^4$.

NUMERICAL VALUE.

57. The **numerical value** of an algebraic expression is the number obtained by substituting for each letter therein its numerical value, and then performing the operations indicated. Thus, if $a = 5$, $b = 4$, and $c = 10$, the numerical value of $4a^2 - bc$ is $4 \times 5^2 - 4 \times 10 = 100 - 40 = 60$.

58. In finding the numerical value of algebraic expressions, the following facts should be kept in mind:

I. *A term preceded by no sign is positive* (+).

II. *The coefficient 1 is understood when no other is given.*

III. *A letter has the exponent 1 when no other is given.*

Let $a = 5$, $b = 2$, $c = 3$, and $d = 1$, in each of the following expressions, and then find their numerical value:

1. $7a - 6b$.
2. $5a + 5bc$.
3. $ab - bc$.
4. $3ab + 5c$.
5. $abc + bc - ac$.
6. $a^2 - b^2c - c^2$.
7. $a^2 + b^2 + c^2$.
8. $3ab^2 - 5b^2c$.
9. $4a - b^2c^2 + 5c$.
10. $5ab^2 + bc - 3bcd$.
11. $4(a + b) + 3(a - b)$.
12. $(a + b)(a - c)$.
13. $5ac + b - 3(b + c)$.
14. $6\left(\frac{2a}{5} + \frac{bc}{3}\right) - bc$.
15. $4(a + b)(c - 2)$.
16. $10(a + b - c) - a(c - b)$.
17. $6(a^2 + b^2 + c) - a^2b^2$.
18. $\frac{ab}{b + c} + \frac{bc}{2} + \frac{a + c}{b}$.
19. $(b^2 + c^2 + d^2) \div \frac{c^2 + d^2}{a}$.
20. $(b + c)^2 - \sqrt{a + b^2}$.
21. $(a + d)^2 - (b + c)^2$.
22. $b^4 + 2b^2c^2 + c^4$.
23. $(a + b)a - b(c + d) - (b - c)d$.
24. $4ab^2 - b^2c^2 + 4(a^2 + b^2 + 2c + d)$.
25. $(a + 2b + c)(a^2 - 2bd) \div 3b^2c$.

POSITIVE AND NEGATIVE NUMBERS.

59. In both arithmetic and algebra the signs + and − are used to denote respectively addition and subtraction, as already shown (§§ 16, 17); but in algebra, as well as in the arts and sciences, these signs are also used to denote that numbers belong to opposite series, that is, are *opposite in quality.* When the signs + and − are thus used, they are called **signs of opposition,** or, better, **signs of quality.**

60. Numbers may be used in opposition to denote condition, motion, direction, time, temperature, value, etc.

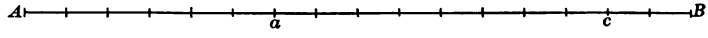
Thus, if north latitude is regarded as +, south latitude is −; if east longitude is +, west longitude is −.

If degrees of temperature above zero are +, degrees of temperature below zero are −.

If distance in a given direction is +, distance in the opposite direction is −.

If profit or gain is +, loss is −; if credits are +, debits are −.

61. The meaning of the signs + and −, when used to denote quality, may be illustrated by movements in opposite directions.



Thus, if distance from any point in the line AB to the *right,* or towards B, is considered +, distance from any point in the line to the *left,* or towards A, is considered −.

If, for example, we start at a and move 8 spaces towards B, the distance is denoted by $+8$; but if we begin at a or at c and move 5 spaces towards A, the distance is denoted by -5.

Thus, if a man starts at a given point and walks in a straight line to the right 8 miles, and then, turning, walks back 5

miles, the distance walked *from* the starting point is denoted by $+8$ miles, and the return distance by -5 miles, and his distance from the starting point is expressed by $+8+(-5) = +8-5 = +3$. He is 3 miles *to the right* of the starting point.

But if, on turning back, the man should walk 12 miles to the left, the return distance would then be denoted by -12 miles, and his position would be expressed by $+8+(-12) = +8-12 = -4$. He would be 4 miles *to the left* of the starting point.

1. A man starts at a given point and walks 4 miles to the right, and then turns and walks 7 miles to the left. Where is he?

2. A ship starts at the equator and sails 50° north, and then turns and sails 35° south. What is its latitude?

3. A ship that is 30° north latitude sails 45° south. What is its latitude?

4. The temperature at noon of a certain day was 60° above zero, and in three hours it fell 25°. What was the temperature at 3 P.M.?

5. The temperature at 10 A.M. was 24° above zero, and at 4 P.M. it had fallen 40°. What was the temperature at 4 P.M.?

6. The temperature at noon of a certain day was $+30°$, and at midnight it was $-15°$. What was the difference in temperature?

62. It is thus seen that in algebra the signs $+$ and $-$ have two distinct uses: one to indicate an *operation*, and the other to denote the *quality* of numbers.

When the sign $+$ or $-$ is placed between two numbers or terms, it indicates an operation; but when the sign $+$ or $-$ is placed before a monomial or before the first term of a polynomial, it is a sign of quality. Thus, in $-ab$, $-a^2+bc$, and $-a^2+ab+b^2$, the sign $-$ denotes quality.

63. The sign between any two terms of a polynomial may be made a sign of quality by preceding it by the sign +. Thus, $a^2 - 2ab + b^2$ may be written $a^2 + (-2ab) + (+b^2)$; that is, $a^2 - 2ab + b^2$ expresses the *sum* of the terms $+a^2$, $-2ab$, $+b^2$.

64. The sign + or − before a parenthesis is a sign of operation when it is preceded by a term, as in $a - (b + c)$; but it is a sign of quality when the parenthesis stands alone, as in $-(a - b)$, or is the first term of a polynomial, as in $-(a - b) + ab$. The first term *within* a parenthesis has the sign + or − as a sign of quality, as in $(-a + bc)$. The sign + is usually not expressed, as in $a - (b + c)$.

65. A number preceded by the sign +, expressed or understood, is called **positive**; and a number preceded by the sign − is called **negative.** When no sign precedes an algebraic number, the sign + is understood (§ 58).

66. The signs + and − are called respectively **positive** and **negative.** When two or more algebraic numbers have each the sign + or −, they are said to have **like signs**; and when one of two algebraic numbers is preceded by + and the other by −, they are said to have **unlike signs.**

67. The value of an algebraic number, considered independently of its sign, is called its **absolute value.** Thus, + 8 and − 8 have the same absolute value, but + 8 has a greater algebraic value than − 8.

The algebraic value of negative numbers decreases as their absolute value increases. Thus, $-10 < -1$, and $-5a < -a$.

Every algebraic number has an absolute value. The sign + or − shows that this value is positive or negative.

68. The range of algebraic numbers is double that of the numbers in arithmetic, since the latter have no sign of quality, the signs + and − being used in arithmetic to denote operations only.

This double range of algebraic numbers is shown by the series:

$$-8\ -7\ -6\ -5\ -4\ -3\ -2\ -1\ -0\ +1\ +2\ +3\ +4\ +5\ +6\ +7\ +8\cdots$$

The numbers on the right of 0 in this series *increase* from left to right ($+4 < +5$), while those on the left of 0 *decrease* from right to left ($-4 > -5$).

LAWS OF THE SIGNS.

69. The manner in which positive and negative numbers are added is shown by the following equations:

$$(1)\quad \begin{cases} +5+(+3) = +5+3 = +8. \\ +2a+(+a) = +2a+a = +3a. \end{cases}$$

$$(2)\quad \begin{cases} +5+(-3) = +5-3 = +2. \\ +2a+(-a) = +2a-a = +a. \end{cases}$$

$$(3)\quad \begin{cases} -5+(+3) = -5+3 = -2. \\ -2a+(+a) = -2a+a = -a. \end{cases}$$

$$(4)\quad \begin{cases} -5+(-3) = -5-3 = -8. \\ -2a+(-a) = -2a-a = -3a. \end{cases}$$

70. These equations may be explained by taking the double scale of numbers, and counting as indicated by the signs.

$$-8\ -7\ -6\ -5\ -4\ -3\ -2\ -1\ -0\ +1\ +2\ +3\ +4\ +5\ +6\ +7\ +8$$

Thus, in (1) begin at $+5$ and count 3 spaces to the right, to $+8$, thus showing that $+5+3=+8$. In (2) begin at $+5$ and count 3 spaces to the left, to $+2$, thus showing that $+5-3=+2$. In (3) begin with -5 and count 3 spaces to the right, to -2, thus showing that $-5+3=-2$. In (4) begin with -5 and count 3 spaces to the left, to -8, thus showing that $-5+(-3)=-5-3=-8$.

71. Since the foregoing explanations are not dependent upon the particular numbers 5 and 3, or the relation of the numbers

$2a$ and a, we may deduce the following laws of the signs in addition:

I. Numbers with like signs, as in (1) and (4) in § 69, are added *by finding the sum of their absolute values, and prefixing the common sign to the result.*

II. Two numbers with unlike signs, as in (2) and (3), are added *by finding the difference of their absolute values, and prefixing the sign of the number that has the greater absolute value.*

More than two numbers with unlike signs may be added by finding the sum of the positive numbers and the sum of the negative numbers, and then adding the two results, as in II.

It is thus seen that the algebraic sum of two numbers with unlike signs is their arithmetical difference with the sign of the greater number.

72. The following equations show the manner in which positive and negative numbers are subtracted:

$$(1)\quad \begin{cases} +5-(+3)=+5-3=+2. \\ +2a-(+a)=+2a-a=+a. \end{cases}$$

$$(2)\quad \begin{cases} +5-(-3)=+5+3=+8. \\ +2a-(-a)=+2a+a=+3a. \end{cases}$$

$$(3)\quad \begin{cases} -5-(+3)=-5-3=-8. \\ -2a-(+a)=-2a-a=-3a. \end{cases}$$

$$(4)\quad \begin{cases} -5-(-3)=-5+3=-2. \\ -2a-(-a)=-2a+a=-a. \end{cases}$$

73. To explain these equations, begin in (1) with $+5$ in § 70, and count 3 spaces to the left, to $+2$, thus showing that $+5-3=+2$. In (2) begin with $+5$, and *since* $-(-3)$ *indicates direction opposite to* -3, count 3 spaces to the right, to $+8$, thus showing that $+5-(-3)=+5+3=+8$. In (3) begin with -5 and count 3 spaces to the left, to -8, thus showing that $-5-3=-8$. In (4) begin with -5, and, since $-(-3)$ indicates direction opposite to -3, count 3 spaces to the right, to -2, thus showing that $-5-(-3)=-5+3=-2$.

74. It is thus shown that numbers with like or unlike signs are subtracted *by changing the sign of the subtrahend, and then adding the resulting numbers.*

75. Since multiplication is the process of taking one number as many times as there are units in another number, the multiplication of positive and negative factors may be shown as follows:

(1) $\begin{cases} +5 \times (+3) = 5+5+5 = +15; \\ +a \times (+b) = a+a+a+\cdots \text{ to } b \text{ terms} = +ab. \end{cases}$

(2) $\begin{cases} -5 \times (+3) = -5-5-5 = -15; \\ -a \times (+b) = -a-a-a-\cdots \text{ to } b \text{ terms} = -ab. \end{cases}$

Since $-3 = -1-1-1$ and $-1 \times 8 = -8 \times 1 = -8$,

(3) $\begin{cases} +5 \times (-3) = -5-5-5 = -15; \\ +a \times (-b) = -a-a-a-\cdots \text{ to } b \text{ terms} = -ab. \end{cases}$

Again, since $-5 \times (-3) = -(-5)-(-5)-(-5) = +5+5+5$,

(4) $\begin{cases} -5 \times (-3) = +5+5+5 = +15; \\ -a \times (-b) = +a+a+a+\cdots \text{ to } b \text{ terms} = +ab. \end{cases}$

76. It is seen from (1) and (4) that the product of two factors with *like* signs is *positive,* and from (2) and (3) that the product of two factors with *unlike* signs is *negative;* that is, in multiplication,

Like signs give $+$, *and unlike signs give* $-$.

77. Since $5 \times 3 = 15$, $15 \div 3 = 5$, and $15 \div 5 = 3$; and since $a \times b = ab$, $ab \div b = a$, and $ab \div a = b$. It is thus seen that division is the inverse of multiplication, and hence the division of positive and negative numbers may be shown as follows:

(1) $+a \times (+b) = ab$; $\therefore +ab \div (+a) = +b$.

(2) $-a \times (+b) = -ab$; $\therefore -ab \div (-a) = +b$.

(3) $+a \times (-b) = -ab$; $\therefore -ab \div (+a) = -b$.

(4) $-a \times (-b) = +ab$; $\therefore +ab \div (-a) = -b$.

78. It is seen from (1) and (2) in § 77 that the quotient obtained by the division of two numbers with *like* signs is *positive*, and from (3) and (4) that the division of two numbers with *unlike* signs is *negative;* that is, in division,

Like signs give +, *and unlike signs give* −.

EXERCISES.

Find the algebraic sum of

1. 15 and 12; 15 and −12; −15 and 12; −15 and −12.

2. 8 + 6 and 5; 8 − 6 and 5; 8 and 6 − 5; −8 and 6 − 5; 8 + 6 and 6 − 5.

3. 12, −7, and −16; 36, −8, −20, and 15.

4. 73, −85, +16, +5; 80, −25, −13, −16; −20, −15, +16; 25, −13, −8, +10.

5. $5a$, $-2a$, $+3a$, $4a$; $16a$, $-7a$, $-3a$, $-2a$; $6a$, $-5a$, $+4a$, $-3a$, $+7a$.

Make the first number given the minuend, and find the algebraic difference of

6. 12 and +5; 12 and −5; −12 and +5; −12 and −5; 5 and −12.

7. 23 and 8; 23 and −8; −23 and 8; −23 and −8; −8 and +23; −8 and −23.

8. −42 and −16; 42 and +16; −42 and 16; 42 and 16; −16 and −42; +16 and +42.

9. $12a$ and $5a$; $12a$ and $-5a$; $-12a$ and $5a$; $-12a$ and $-5a$; $+5a$ and $-12a$.

10. $15a$ and $-7a$; $7a$ and $15a$; $-7a$ and $15a$; $-7a$ and $-15a$; $+7a$ and $+15a$.

Multiply

11. 27 by -4; -27 by 4; -27 by -4.

12. $3a$ by 9; $4a$ by -15; $-3a$ by 12; $-5a$ by -13; $-12a$ by 5; $12a$ by -3.

13. $36a$ by 3; $36a$ by -3; $-36x$ by 3; $-36x$ by -2; 36 by $-2x$.

14. $6ab$ by 5; $6ab$ by -5; $-5ab$ by 12; $-5ab$ by -25; 25 by $-3ab$; -25 by $4ab$.

Divide

15. 60 by -12; -72 by 8; -84 by -12.

16. $15x$ by -5; $-15x$ by -3; $-40x$ by 8.

17. $21x$ by $3x$; $42x$ by $-7x$; $-18x$ by $-9x$.

18. $15ax$ by -5; $-33ax$ by 11; $-48ax$ by $-16ax$; $+48x$ by $-12x$.

LAWS OF OPERATION.

79. Algebraic operations involve several fundamental principles of so wide application that they are called **Laws of Operation.**

These laws may be clearly indicated by first using particular numbers, represented by figures, and then using letters denoting general numbers.

80. I. The Commutative Law.

Sum. $\begin{cases} 7+3=3+7;\ 7+3-5=3-5+7;\ \text{etc.} \\ a+b=b+a;\ a+b-c=b-c+a;\ \text{etc.} \end{cases}$

Product. $\begin{cases} 7\times 3=3\times 7;\ 7\times 3\times 5=3\times 5\times 7;\ \text{etc.} \\ a\times b=b\times a;\ a\times b\times c=b\times c\times a;\ \text{etc.} \end{cases}$

The principle thus indicated is called the **Commutative Law.** It may be stated as follows:

The sum or the product of two or more numbers is the same in whatever order the numbers are taken.

78. It is seen from (1) and (2) in § 77 that the quotient obtained by the division of two numbers with *like* signs is *positive*, and from (3) and (4) that the division of two numbers with *unlike* signs is *negative;* that is, in division,

Like signs give +, *and unlike signs give* −.

EXERCISES.

Find the algebraic sum of

1. 15 and 12; 15 and -12; -15 and 12; -15 and -12.

2. $8+6$ and 5; $8-6$ and 5; 8 and $6-5$; -8 and $6-5$; $8+6$ and $6-5$.

3. 12, -7, and -16; 36, -8, -20, and 15.

4. 73, -85, $+16$, $+5$; 80, -25, -13, -16; -20, -15, $+16$; 25, -13, -8, $+10$.

5. $5a$, $-2a$, $+3a$, $4a$; $16a$, $-7a$, $-3a$, $-2a$; $6a$, $-5a$, $+4a$, $-3a$, $+7a$.

Make the first number given the minuend, and find the algebraic difference of

6. 12 and $+5$; 12 and -5; -12 and $+5$; -12 and -5; 5 and -12.

7. 23 and 8; 23 and -8; -23 and 8; -23 and -8; -8 and $+23$; -8 and -23.

8. -42 and -16; 42 and $+16$; -42 and 16; 42 and 16; -16 and -42; $+16$ and $+42$.

9. $12a$ and $5a$; $12a$ and $-5a$; $-12a$ and $5a$; $-12a$ and $-5a$; $+5a$ and $-12a$.

10. $15a$ and $-7a$; $7a$ and $15a$; $-7a$ and $15a$; $-7a$ and $-15a$; $+7a$ and $+15a$.

Multiply

11. 27 by -4; -27 by 4; -27 by -4.

12. $3a$ by 9; $4a$ by -15; $-3a$ by 12; $-5a$ by -13; $-12a$ by 5; $12a$ by -3.

13. $36a$ by 3; $36a$ by -3; $-36x$ by 3; $-36x$ by -2; 36 by $-2x$.

14. $6ab$ by 5; $6ab$ by -5; $-5ab$ by 12; $-5ab$ by -25; 25 by $-3ab$; -25 by $4ab$.

Divide

15. 60 by -12; -72 by 8; -84 by -12.

16. $15x$ by -5; $-15x$ by -3; $-40x$ by 8.

17. $21x$ by $3x$; $42x$ by $-7x$; $-18x$ by $-9x$.

18. $15ax$ by -5; $-33ax$ by 11; $-48ax$ by $-16ax$; $+48x$ by $-12x$.

LAWS OF OPERATION.

79. Algebraic operations involve several fundamental principles of so wide application that they are called **Laws of Operation.**

These laws may be clearly indicated by first using particular numbers, represented by figures, and then using letters denoting general numbers.

80. I. The Commutative Law.

Sum. $\begin{cases} 7+3=3+7;\ 7+3-5=3-5+7;\ \text{etc.} \\ a+b=b+a;\ a+b-c=b-c+a;\ \text{etc.} \end{cases}$

Product. $\begin{cases} 7\times 3=3\times 7;\ 7\times 3\times 5=3\times 5\times 7;\ \text{etc.} \\ a\times b=b\times a;\ a\times b\times c=b\times c\times a;\ \text{etc.} \end{cases}$

The principle thus indicated is called the **Commutative Law.** It may be stated as follows:

The sum or the product of two or more numbers is the same in whatever order the numbers are taken.

81. It follows that the terms of a polynomial or the factors of a product may be arranged in any order, provided the signs of the terms or factors are not changed. Thus,

$x^2 - 2xy + y^2 = x^2 + y^2 - 2xy$, and $3 \times x \times y = y \times x \times 3$.

82. II. The Associative Law.

Sum.

$$\begin{cases} (12+5)+7 = 12+(5+7); \ (8+6)-3 = 8+(6-3), \text{ etc.} \\ (a+b)+c = a+(b+c); \ (a+b)-c = a+(b-c), \text{ etc.} \end{cases}$$

Product.

$$\begin{cases} (8\times 6)\times 5 = 8\times(6\times 5); \ 8\times(6\times 5\times 4) = (8\times 6)\times(5\times 4), \text{ etc.} \\ (a\times b)\times c = a\times(b\times c); \ a\times(b\times c\times d) = (a\times b)\times(c\times d), \text{ etc.} \end{cases}$$

The principle thus indicated is called the **Associative Law.** It may be thus stated:

The sum or the product of three or more numbers is the same in whatever way the numbers may be grouped.

This law is the same in principle as the Commutative Law.

83. III. The Distributive Law.

Product.

$$\begin{cases} 3(2+3+5) = 3\cdot 2 + 3\cdot 3 + 3\cdot 5 = 6+9+15; \\ 4(4+3-5) = 4\cdot 4 + 4\cdot 3 - 4\cdot 5 = 16+12-20; \\ a(b+c+d) = ab+ac+ad; \ (a+b-c)x = ax+bx-cx. \end{cases}$$

Quotient. $\dfrac{5+10}{3} = \dfrac{5}{3} + \dfrac{10}{3}$; $\dfrac{b+c+d}{a} = \dfrac{b}{a} + \dfrac{c}{a} + \dfrac{d}{a}$.

The principle thus indicated is called the **Distributive Law.** It may be thus stated:

I. *The sum of several numbers multiplied by a given number equals the sum of the products of the several numbers multiplied by the given number.*

II. *The sum of several numbers divided by a given number equals the sum of the quotients of the several numbers divided by the given number.*

84. IV. The Exponent Law.

$5^2 = 5 \cdot 5$; $5^3 = 5 \cdot 5 \cdot 5$. $\therefore 5^2 \times 5^3 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^5 = 5^{2+3}$.

$a^2 = aa$; $a^3 = aaa$. $\therefore a^2 \times a^3 = aaaaa = a^5 = a^{2+3}$.

$a^m = aaa \cdots$ to m factors; $a^n = aaa \cdots$ to n factors. Hence

$$a^m \times a^n = aaa \cdots \text{ to } (m+n) \text{ factors} = a^{m+n}.$$

The principle thus indicated is called the **Exponent Law.** It may be stated as follows:

The exponent of the product of two or more powers of a number is equal to the sum of the exponents of the given powers. Hence

The product of the several powers of a number is found *by adding their exponents.* Thus, $x^2 \times x^6 = x^{2+6} = x^8$.

The above law is commonly called the *Index Law;* but in this treatise the symbol that denotes the *powers* of numbers is called an *exponent*, and the symbol that denotes the *roots* of numbers an *index*.

EQUATIONS AND PROBLEMS.

85. The value of x found in the solution of an equation may be **verified** by substituting such value for x in the original equation. If the two members are equal, the value of x found is correct.

Thus, in solving the equation $5x - 2x + 4 = 25$, the value of x found is 7. Substituting 7 for x in the equation, we have $5 \times 7 - 2 \times 7 + 4 = 25$, or $35 - 14 + 4 = 25$, or $25 = 25$, and hence 7 is the correct value of x.

Find and verify the value of x in the following equations:

1. $7x + 5x - 4x = 24$.
2. $13x - 6x + 8x = 45$.
3. $15x - 6x - 4x = 30$.
4. $20x - 12x - 3x = 40$.
5. $8x + (6x - 7x) = 56$.
6. $10x - (3x + 2x) = 35$.
7. $12x - (7x - 3x) = 32$.
8. $2(9x + 7x) - (3x + 5x) = 48$.
9. $15x - (4x + 6x - 3x) = 48$.
10. $3(x + 4) + 5x = 36$.

11. $7x + 5(x + 2) = 34$.
12. $12x - 4(x + 5) = 36$.
13. $9x - 5(x + 1) = 19$.
14. $13x - 5(x - 3) = 55$.
15. $8(x - 3) - 5x + 24 = 33$.
16. $5(2x - 5) - 4(x - 5) = 37$.
17. $6(x - 3) + 3(x + 6) = 27$.
18. $6(x+4) - (3x+6\times3) = 36$.
19. $5(2x-4) - (6\times5-2x) = 70$.
20. $4(3x-5) + 3(x-4) = 13$.

21. A farmer raised 720 bushels of grain, consisting of wheat, corn, and oats. He raised twice as much wheat as corn, and 3 times as much oats as corn. How many bushels of each grain did he raise?

22. Divide \$126 into three parts such that the second part shall be twice the first, and the third part 3 times the second.

23. A market woman has 4 times as many dimes as quarters, and twice as many nickels as dimes, and in all she has 52 pieces of money. How many pieces of each kind has she?

24. In three years a merchant made a profit of \$6300. The profit the second year was twice the profit of the first year, and the profit the third year was twice that of the second year. What was the profit each year?

25. Three men, A, B, and C, form a partnership in business, with a capital of \$12,000. A furnished twice as much as B, and C as much as A and B together. How much capital did each furnish?

26. The difference of two numbers is 18, and one of the numbers is 3 times the other. What are the numbers?

27. A has \$21 more than B, and A's money is 4 times B's. How much money has each?

28. A father is twice as old as his elder son, and the elder son is 3 times as old as the younger, and the sum of all their ages is 80 years. How old is each?

29. A newsboy, in counting his week's earnings, found that he had twice as many dimes as quarters, and three times as

many nickels as dimes, and that he had in all $7.50. How many pieces of money had he of each kind?

Let x = number of quarters; then $25x$ = their value;
$2x$ = number of dimes, and $20x$ = their value;
$6x$ = number of nickels, and $30x$ = their value.
Hence $25x + 20x + 30x = 75x = 750$.

30. A newsboy has $4.80 in quarters, dimes, and nickels, and of each an equal number. How many pieces of money has he? How much money in quarters, dimes, and nickels respectively?

31. A man and his two sons earn $162 a month, and the man earns 3 times as much as the elder son, and the elder son twice as much as the younger. How much does each earn?

32. Twice A's age is 20 years more than B's age and 10 years more than C's age, and the sum of their ages is 120 years. What is the age of each?

33. A man owns two farms which together contain 180 acres of land, and the first farm contains 20 acres more than the second. How many acres in each farm?

34. A farm of 640 acres was divided among a brother and two sisters. The two sisters received an equal share, and the brother received 40 acres less than the number of acres received by the two sisters together. How many acres did each receive?

35. A and B together have $180, and B has $20 less than 3 times A's money. How much money has each?

36. A, B, and C are partners in business. A's capital is twice B's, and C's capital is $500 less than 3 times B's, and the total capital is $14,020. How much capital has each?

37. Two men start at the same time from two places which are 63 miles apart, and travel towards each other, one at the rate of 3 miles an hour, and the other 4 miles an hour. In how many hours will they meet? How far will each travel?

SUGGESTION. Let x = the number of hours.

38. Three military companies muster together 195 men. The second company musters 30 men more than the first, and the third musters one half as many men as the first and second together. How many men in each company?

39. A lady bought 18 yards of silk and 15 yards of serge for $38.25, and the silk cost twice as much per yard as the serge. What was the cost of each per yard?

40. A fruit dealer sold several dozens of oranges at 25 cents a dozen, twice as many lemons at 15 cents a dozen, and twice as many pears as lemons at 10 cents a dozen, and the bill for all was $3.80. How many dozens of each kind of fruit did he sell?

41. A man bought a suit of clothes for $30. The coat cost $5 more than the trousers, and the trousers twice as much as the vest. What was the cost of each?

42. A woman bought a cloak, dress, and bonnet for $35. The dress cost $8 more than the bonnet, and the cloak $4 more than the dress. What was the cost of each?

43. A jeweler sold three watches for $100. He sold the second watch for $15 more than the first, and the third watch for $5 less than the second. How much did he receive for each watch?

44. Two boys, Charles and Harry, had an equal amount of money. Charles paid 75 cents for a book, and Harry 50 cents for a knife, and then Harry had twice as much money as Charles. How much money had each at first?

45. A farm containing 160 acres was divided among three heirs, A, B, and C. A received 20 acres more than B, and C received as many acres as both A and B less 40 acres. How many acres did each heir receive?

46. Two railroad trains start from two cities 495 miles apart, and run towards each other on the same track, one running 30 miles an hour and the other 25 miles an hour. In how many hours will they meet?

CHAPTER III.

ADDITION AND SUBTRACTION.

ADDITION.

86. Addition in algebra is the process of combining two or more given numbers into one number, called the **sum**.

Add

1. $3a$, $5a$, $2a$, and $7a$.
2. $-2a$, $-5a$, $-2a$, and $-7a$.
3. ab, $4ab$, $3ab$, and $5ab$.
4. $-ab$, $-4ab$, $-6ab$, and $-7ab$.
5. $4a$, $6a$, $-5a$, and $-7a$.
6. $3ax$, $-5ax$, $-4ax$, and $8ax$.
7. a, b, and c.

SUGGESTION. The sum of a, b, and c is $a + b + c$.

8. a, $3b$, and $-d$.
9. $2ab$, $3ab$, and $-13c$.
10. $3x$, $-x$, $4y$, and $-y$.

87. The first six of the foregoing examples show that *like numbers* can be combined into one term, and the last four show that *unlike numbers* can only be added by connecting them with the proper sign.

88. These examples also show that algebraic numbers may be

I. *Like numbers with the same sign.*
II. *Like numbers with unlike signs.*
III. *Unlike numbers, i.e., no two numbers like.*
IV. *Some numbers like, and others unlike.*

89. These facts give four classes of examples in the addition of monomials.

I. Like numbers with the same sign.

Add

11. $7\,bc$, $9\,bc$, bc, $5\,bc$.

12. $-7\,bc$, $-9\,bc$, $-bc$, $-5\,bc$.

13. $-3\,xy$, $-xy$, $-10\,xy$, $-4\,xy$, $-7\,xy$.

14. $-12\,abc$, $-15\,abc$, $-abc$, $-2\,abc$.

15. $3\,a^2b$, a^2b, $12\,a^2b$, $5\,a^2b$, $13\,a^2b$.

16. $-12\,cx^2$, $-cx^2$, $-2\,cx^2$, $-13\,cx^2$, $-9\,cx^2$.

17. $-5\,(a+b)$, $-6\,(a+b)$, $-3\,(a+b)$, and $-(a+b)$.

18. $5xy\,(a-b)$, $3xy\,(a-b)$, $4xy\,(a-b)$, and $xy\,(a-b)$.

19. $\sqrt{ab}$, $4\sqrt{ab}$, $6\sqrt{ab}$, $9\sqrt{ab}$, and $12\sqrt{ab}$.

20. $-4\,mn\sqrt{x^2}$, $-3\,mn\sqrt{x^2}$, $-9\,mn\sqrt{x^2}$, and $-mn\sqrt{x^2}$.

90. To add like algebraic numbers with the same sign,

Add the coefficients, and to their sum annex the literal part, and prefix the common sign.

II. Like numbers with unlike signs.

21. Add $7\,ab$, $6\,ab$, $-2\,ab$, $-5\,ab$, $3\,ab$.

$$\begin{array}{r} 7\,ab \\ 6\,ab \\ -2\,ab \\ -5\,ab \\ 3\,ab \\ \hline 9\,ab \end{array}$$

For convenience write the numbers to be added in a column, as at the left. Add the positive numbers and the negative numbers, and to the arithmetical difference of the coefficients of their sums annex the literal part, and prefix the sign of the greater. Thus, $7\,ab + 6\,ab + 3\,ab = 16\,ab$; and $-2\,ab - 5\,ab = -7\,ab$. $16\,ab - 7\,ab = 9\,ab$.

It is unnecessary to write the numbers to be added in a column when the coefficients can be readily added. If preferred, the positive and negative monomials may be written in separate columns.

Add

22. $7\,ax, -2\,ax, -5\,ax, 6\,ax, 3\,ax.$

23. $-3\,ac, -ac, -5\,ac, -3\,ac, 7\,ac, 12\,ac.$

24. $-9\,by, -by, 5\,by, by, 3\,by, 2\,by.$

25. $5\,abx, -2\,abx, -3\,abx, 10\,abx, -4\,abx.$

26. $3\,xz^2, xz^2, -7\,xz^2, 10\,xz^2, -xz^2.$

27. $8\,(a+c), -3\,(a+c), -11\,(a+c), 7\,(a+c), 4\,(a+c).$

28. $13\,(x+y), 5\,(x+y), -12\,(x+y), 9\,(x+y), -15\,(x+y).$

29. $15\sqrt{x}, \sqrt{x}, -13\sqrt{x}, 3\sqrt{x}, -5\sqrt{x}.$

30. $10\,a\sqrt{xy}, 9\,a\sqrt{xy}, -15\,a\sqrt{xy}, -7\,a\sqrt{xy}, 8\,a\sqrt{xy}.$

91. To add like numbers with unlike signs,

Add the positive numbers and the negative numbers, and to the arithmetical difference of their sums prefix the sign of the greater.

III. Unlike numbers.

92. In arithmetic the expression $7+5$ indicates that the two numbers are to be added, and the result (12) is their sum; but in algebra the expression $a+b$ not only indicates that a and b are to be added, but it also represents their sum (§ 21). Hence, when the algebraic numbers to be added are unlike, their sum is expressed by connecting them with their proper signs in the form of a polynomial. Thus, the sum of a, b, and $-c$ is $a+b-c$.

Every polynomial may be regarded as the expressed sum of its several terms. Thus, $ab^2 - bc^2 + ac^2$ is the sum of ab^2, $-bc^2$, and ac^2.

31. What is the sum of x, y, and $-z$?

32. Add $3\,ab^2$, $-4\,a^2b$, and $-5\,c$.

33. Add x^2, $-2\,xy$, and y^2.

34. Add $2\,x^3$, $-3\,x^2y$, $-4\,xy^2$, and $5\,y^3$.

35. Add ax, bx, and $-cx$, and reduce the sum to its simplest form.

SUGGESTION. $ax + bx - cx = (a + b - c)x$.

36. Add $3ax^2$ and $-9axy$, and reduce to simplest form.

93. To add unlike numbers,

Connect the numbers to be added with their signs, and reduce the result to its simplest form.

In algebra the results of given operations are often indicated by their proper signs.

IV. Some Numbers Like, and Others Unlike.

37. Add $15a$, $3bc$, $-7a$, $-3a$, $-10bc$, $-2d$, and 3.

Process.

$$\begin{array}{r} 15a \\ -\ 7a \quad 3bc \\ -\ 3a - 10bc - 2d + 3 \\ \hline 5a - \ 7bc - 2d + 3 \end{array}$$

In the numbers to be added, $15a$, $-7a$, and $-3a$ are like numbers, and their algebraic sum is $5a$; $3bc$ and $-10bc$ are like numbers, and their algebraic sum is $-7bc$. The sum sought is $5a - 7bc - 2d + 3$, written in the form of a polynomial.

Add

38. $3ab$, $5a^2b$, $6ab$, $-8a^2b$, $-a^2b$, $6ab$, and $2ab^2$.

39. $-3ax$, $-5ax^2$, $7ax$, $-3ax$, $7ax^2$, $-x^2y^2$, and $4ax$.

40. $3\sqrt{xy}$, $-6x^2y$, $-8\sqrt{xy}$, $5\sqrt{xy}$, and $12x^2y$.

41. x^3, $-2x^2y$, $-xy^2$, $5x^2y$, $4xy^2$, and y^3.

42. $3anx^2$, $-7nx^3$, $-5anx^2$, $4nx^3$, and $10anx^2$.

43. $15ab$, $-6b$, $-7ab$, $10b$, and $3c$.

44. $12ab$, $-8bc$, $-5ab$, $13bc$, and $-3abc$.

45. $23x^2y$, $-14xy^2$, $-17x^2y$, $20xy^2$, and $-3x^2y$.

94. To add numbers, some like and others unlike,

Add the like numbers, and connect their sums and the unlike numbers with their proper signs.

Polynomials.

1. Add $8ax - 3cy$, $3ax - 7cy$, $4ax - cy - 9ab$, and $9cy + 4ab + ax$.

Process.

$$\begin{array}{r} 8ax - 3cy \\ 3ax - 7cy \\ 4ax - cy - 9ab \\ ax + 9cy + 4ab \\ \hline 16ax - 2cy - 5ab \end{array}$$

Write the like terms in columns, as at the left, and add as in § 94. The terms of a polynomial may be arranged in any order (Commutative Law, § 80).

Add

2. $5a + b - c$, $2a - 10b + 3c$, $8a + 7b - 8c$, $a + 2b + 6c$.

3. $a + b + c$, $a - b + c$, $a + b - c$, $a - b - c$, $b - a + c$.

4. $4x + 5y - 6z$, $4y - 6x + 2z$, $x - y - z$, $x + 3y + 3z$.

5. $5x^2 - y^2 + z^2$, $y^2 - 2x^2 + 3z^2$, $x^2 - 5y^2 + 2z^2$, $-3x^2 + 5y^2 - 2z^2$.

6. $3az - 4by - 8$, $-2az + 5by + 6$, $7 + 6by - 5az$, $5by - 8az + 4$, $18 - 3by + 5az$.

7. $8ay^2 - 3cz^2 + 2$, $5ay^2 - 2cz^2 + 5$, $7cz^2 - 10ay^2 - 3$, $7ay^2 + cz^2 + 1$, $5 - 2cz^2 - 6ay^2$.

8. $19a^2x^2 + 8b^2y^2 + 2c^2z^2$, $a^2x^2 - 17b^2y^2 + 5c^2z^2$, $12a^2x^2 + 5b^2y^2 - 10c^2z^2$, $7b^2y^2 - 15a^2x^2 + 3c^2z^2$.

9. $5a^2bc - 7ab^2c + 3abc^2$, $17ab^2c - 10abc^2$, $12ab^2c - 11a^2bc$.

10. $4a^2 - 3c^2 + d^2$, $5b^2 - 2a^2 + c^2 + d^2$, $2c^2 - 7b^2 + 5d^2$.

11. $7y^2 - 5x^2 + abc$, $2x^2 + 9y^2 - 3abc$, $10x^2 + 2abc$, $17x^2 - 4y^2$.

12. $px - qy + m + n$, $5qy - px + 3m - 3n$, $3px - 4qy - 4m + n$.

13. $21x^2 - 35y^2 + 71z^2$, $82z^2 - 17x^2 - 43y^2$, $73y^2 - 52z^2 - 4x^2$.

14. $18ab + 93bc - 101ac$, $83ac - 17ab - 23bc$, $47bc - 7ab + 8ac$, $3ab + 12ac - 32bc$.

15. $121x^2 + 144y^2 + 49z^2$, $25x^2 - 64y^2 - 16z^2$, $9y^2 - 169x^2 - 36z^2$, $42x^2 - 26y^2 - 17z^2$, $12y^2 - 33x^2 - 14z^2$.

16. $12x^2 + pxy - 2y^2$, $13pxy - 11y^2 - 14x^2$, $15y^2 + 26pxy$, $15x^2 - 15pxy$, $10x^2 - 7y^2$.

17. $ax^3 + bx^2 - 3cx$, $2ax^3 + 3bx^2 - 4cx$, $7ax^3 + 4cx - 5bx^2$, $bx^2 - 5cx$, $7cx - 5ax^3$, $3cx - 4bx^2$.

18. $x^3 + 2x^2 + 3x + 4$, $6x - 5x^2 + 3x^3 - 2$, $3x^2 - 4x^3 + 5 - 9x$.

95. A polynomial with two or more terms which have a common factor may be simplified *by writing the factors not common within a parenthesis preceded or followed by the common factor.* Thus, $ax + ay - 2az$ may be written as $a(x + y - 2z)$; $x^3 - 3ax + 5bx$, as $(x^2 - 3a + 5b)x$; and $x^2 - ab + ac^2$, as $x^2 - a(b - c^2)$.

Simplify

19. $mnx^2 - mnx + mny$; $3ax^3 - 3ax^2 - 3axy^3 - 6ay^4$.

20. $x^2 - ax - bx + cx + ab$; $a^2b - a^2c - b^2d + b^2c$.

21. $x^3 - 3x^2y + 5x^2y^2$; $5x^2 - 10y + 6x + 12y^2$.

22. $ax + by - cx - dy$; $a\sqrt{x} - a\sqrt{y} + a\sqrt{2z}$.

SUBTRACTION.

96. Subtraction is the process of taking one number from another.

The result obtained by taking one number from another is called their **difference.**

The number subtracted is called the **subtrahend,** and the number from which the subtrahend is taken is called the **minuend.**

The difference of two numbers is found by subtracting the second from the first, the number first given being the minuend. Thus, the difference of a and b is $a - b$.

97. If either of two numbers be taken from their sum, the result will be the other number. Thus, $7 + 2 = 9$; $9 - 2 = 7$, and $9 - 7 = 2$. It is thus seen that subtraction is the inverse of addition.

98. If the subtrahend and difference be added, their sum will be the minuend. It follows that the minuend may be regarded as the sum, and the subtrahend and difference as the numbers added. Hence

Subtraction may be defined as the process of finding a number which, added to a given number, will equal another given number.

99. In arithmetic the subtrahend is equal to or less than the minuend; and the minuend, subtrahend, and difference are like numbers.

In algebra the subtrahend may be numerically greater than the minuend, as $2 - 7 = -5$; and the minuend and subtrahend may be like or unlike numbers (§ 54).

100. It has been shown in § 74 that a positive number is subtracted by changing its sign to $-$, and a negative number by changing its sign to $+$, and then combining the resulting numbers as in addition.

Monomials.

1. From $5a$ take $2a$; from $5a$ take $-2a$.
2. From $-5a$ take $2a$; from $-5a$ take $-2a$.
3. From $12ab$ take $-5ab$; from $-12ab$ take $7ab$.
4. From $-7ax$ take $-3ax$; from $7ax$ take $-3ax$.
5. From $2xy$ take $10xy$; from $-2xy$ take $-10xy$.
6. From $13ax^2y$ take $-5ax^2y$; from $5ax^2y^2$ take $12ax^2y^2$.
7. From x take y; from x take $-y$.
8. From $3x$ take 5; from $2x$ take -3.

101. To subtract one monomial from another,

Change the sign of the subtrahend, and then proceed as in addition.

With a little practice the sign of the subtrahend can be changed *mentally;* i.e., it may be considered as changed.

Polynomials.

9. From $5ab - 7ax^2 + 3$ take $3ab + 5ax^2 - 7$.

Process.

$$\begin{array}{r} 5ab - 7ax^2 + 3 \\ 3ab + 5ax^2 - 7 \\ \hline 2ab - 12ax^2 + 10 \end{array}$$

For convenience write the subtrahend under the minuend, as at the left. Changing the sign, $3ab$ becomes $-3ab$, and $-3ab$ and $5ab$ added are $2ab$; changing the sign, $+5ax^2$ becomes $-5ax^2$, and $-5ax^2$ and $-7ax^2$ are $-12ax^2$; changing the sign, -7 becomes $+7$, and $+7$ and $+3$ are $+10$. The difference is $2ab - 12ax^2 + 10$.

Proof. The sum of $2ab - 12ax^2 + 10$ and $3ab + 5ax^2 - 7$ is $5ab - 7ax^2 + 3$.

10. From $9xy - 3x^2 - y$ take $-3xy + 5x^2 - 4y$, and prove by addition.

11. From $x^2 - 2xy + y^2$ take $x^2 + 2xy + y^2$, and prove by addition.

12. From $a + b$ take $a - b$, and prove by addition.

13. From $5a^2 - 3ab$ take $a^2 + ab$, and prove by addition

14. From $10a - 5b + 3c$ take $8a - 7b + 5c$.

15. From $15a^2 - 7ab + c$ take $12a^2 + 3ab + 4c$.

16. From $9x^2 - 5xy - 8$ take $4x^2 - 7xy + 2$.

17. From $x^3 + 3x^2y + 3xy^2 + y^3$ take $x^3 - 3x^2y + 3xy^2 - y^3$.

18. From $x^3 - 3x^2y + 3xy^2 - y^3$ take $x^3 + 3x^2y + 3xy^2 + y^3$.

19. From $3a^2 - 5ab - 2b^2$ take $5a^2 - 5ab + b^2 - 10$.

20. From $7x^3 - 2ax^2 + 5a^2x - a^3$ take $5x^3 - 3a^2x - a^3$.

21. From $a + b - c + d - e$ take $a - b - c - d + e$.

22. From $a - b - c - d - e$ take $a - b + c - d - e$.

23. From $x^2 + qx - pv$ take $x^2 + pv + qx$.

24. From $6a^2 + 8ab + 10b^2$ take $5a^2 + 6ab + 7b^2$.

25. From $p^3 - 4p^2x - 5px^2 + x^3$ take $p^3 - 12p^2x + x^3$.

26. From $21x^3 - 41ax^2 + 72a^3$ take $12x^3 - 72ax^2 + 54a^3$.

27. From $x^4 + 4x^3y + 6x^2y^2 + y^4$ take $8x^3y - 6x^2y^2 - y^4$.

28. From $5(x^2+y^2)+3(a^2-b^2)$ take $2(x^2+y^2)-7(a^2-b^2)$.

29. From $6\sqrt{a^2+b^2}+\sqrt{a^2-b^2}+4\sqrt{3}$ take $\sqrt{a^2+b^2}-\sqrt{3}$.

30. From $x^3-3abx^2y+x^2$ take $y^3-3abx^2y+y^2$.

31. From $x^3-3x^2y+3xy^2-y^3$ take $x^3+3xy^2-y^3$.

102. To subtract polynomials,

Write the subtrahend under the minuend so that like terms, if any, shall be in the same column.

Change the signs of the terms of the subtrahend, and then proceed as in addition.

Exercises in Addition and Subtraction.

1. Add $3(x+y)$, $5(x+y)$, and $(x+y)$.

2. Add $5\sqrt{x-y}$, $\sqrt{x-y}$, and $-3\sqrt{x-y}$.

3. Add $a(x^2-y^2)$, $b(x^2-y^2)$, and $2a(x^2-y^2)$.

4. Add $3x^2+\sqrt{a-x}$, $5x^2-\sqrt{a-x}$, and $7x^2-(a-x^2)$.

5. From $2x-3\sqrt{xy}+2y$ take $x-y+2\sqrt{xy}$.

6. From $3a^2+2ax+4x^2$ take a^2-ax-x^2.

7. From x^3+3x^2c+xc take $x^3-(2x^2c-3xc)$.

8. Add $(a-b)x$, $(a-b)y$, and $(a-b)z$.

9. Add $3a(5-x)$, $2a(5-x)-7a(5-x)$, and $a(5-x)$.

10. From the sum of $x^2+2xy+y^2$ and x^2-y^2 take $x^2-2xy+y^2$.

11. From ax^2-bx^2 take the sum of $3ax^2+5bx^2$ and $-5ax^2-4bx^2$.

12. Add $y^4-2x^2y^2+6$, $5y^4+3x^3y$, $3xy^3+2$, and $y^4+2xy^3+3x^2y^2+4x^3y$.

13. Add $3(a+b)+5(a-b)$, $7(a+b)-2(a-b)$, and $12(a+b)-4(a-b)$.

14. Add $6(x^2-y^2)$, $-5(x^2-y^2)$, $-7(x^2-y^2)$, and $4(x^2-y^2)$.

15. Add $8(a^2+b^2)+5(a^2-b^2)$, $2(a^2+b^2)-12(a^2-b^2)$, and $12(a^2+b^2)+8(a^2-b^2)$.

16. Add $5(a+b+c)$, $-7(a+b+c)$, $11(a+b+c)$, and $13(a+b+c)$.

17. From the sum of $5c+3a\sqrt{x^2y^2}$, $-5a\sqrt{x^2y^2}$, and $a\sqrt{x^2y^2}$ take the sum of $a\sqrt{x^2y^2}$ and $3c-2a\sqrt{x^2y^2}$.

18. From $x^3+3x^2y+3xy^2+y^3$ take x^3-y^3, and from the result take $3x^2y-3xy^2$.

Parentheses.

103. The subtraction of one algebraic number from another may be indicated by writing the subtrahend in a parenthesis, and preceding it by the sign $-$. Thus, $x^2+y^2-(x^2-2xy+y^2)$ indicates that $x^2-2xy+y^2$ is to be subtracted from x^2+y^2.

104. If a parenthesis is preceded by the sign $+$, the parenthesis may be removed *without changing any of the signs within it;* but, when a parenthesis is preceded by the sign $-$, *the signs $+$ and $-$ within it must be changed* when the parenthesis is removed (§ 46). Thus, $x^2+y^2-(x^2-2xy+y^2)=x^2+y^2-x^2+2xy-y^2=2xy$.

Conversely, any number of terms of a polynomial may be inclosed in a parenthesis preceded by the sign $+$ *without changing the sign of any term;* but, when the terms of a polynomial are inclosed in a parenthesis preceded by the sign $-$, *the signs of all the terms inclosed must be changed.* Thus,

$$a^3-3a^2b+3ab^2-b^3=a^3-3a^2b+(3ab^2-b^3);$$
$$a^3-3a^2b+3ab^2-b^3=a^3-(+3a^2b-3ab^2+b^3).$$

Remove the parenthesis and collect like terms in

1. $5a+b-(4a+2b)$.
2. $a+b-c-(2a+b+c)$.
3. $a^2+2c^2-(-3a^2+2c^2-ac)$.

4. $a^2 + b^2 - (a^2 - 2ab + b^2) + a^2 - (a^2 + 2ab + b^2)$.

5. $5x - (4x - \overline{3x + 5})$.

SUGGESTION. First remove the vinculum, and then the parenthesis. It is best for the pupil to begin always with the *innermost* sign of aggregation.

6. $x^2 - (2xy + y^2) - (3x^2 - \overline{x^2 - 2xy + y^2})$.

7. $x^2 - (x^2 - \overline{xy - y^2} - y^2 + x^2)$.

8. $12 - x^2 - [-x^2 - (x^2 - x^2 + \overline{5 - x^2})]$.

9. $a^2 + b^2 - [(a^2 + 2ab + b^2 - \overline{a^2 - b^2}) + 2b^2] + 4b^2$.

10. $x^4 + x^3y + 3x^2y^2 - (x^3y + 3x^2y^2 + y^4)$.

11. $3ax^3 - 2bx^2 + 2bx - 4 - (-ax^3 - 3bx^2 + bx - 13)$.

12. $a^2x^2 - ax^3 - x^4 + (3a^2x^2 + 2ax^3 + x^4)$.

13. $a^2b^2 - (ab - b^2) + 3c + (a^2b^2 - ab - b^2 - 3c)$.

14. $a + b + c - (a + b - c) - (a - b + c) - (c - 2a + b)$.

15. $2a - (3b + c) + [c - (a - 2b)]$.

16. $5x - [y + (4x - y)] - (x - y)$.

17. $3m - n - [p - (2m - 7n)]$.

18. $8m - [4n - (4m - 4n - r)]$.

19. $a - (2b - c) + 3c - (4b - 2c) - (2a + b + 4c)$.

20. $3x - [-4x + (3x + 5) - (2x - 6)]$.

21. $x + y - [(3x - y) + (x - 3y) - x] + [x - (y - 2x)]$.

22. $2a - [3b - (4a - 5b)] - (6a - 7b)$.

23. $(a - b) - \{c - [d - (a + b) + c] - (a + b)\}$.

24. $ab - \{-bc - [a - (b - c) - 2ab - (a - b)]\}$.

25. $5x - \{5y - [x - (3z - 3y) + 2z - (5x - 2y - z)]\}$.

26. $1 - [1 - a - (a^2 + a - 1) - (a - 2a^2 - 1)]$.

27. $a - b - \{-3c - [(a + b) - (a - b - c)] - 3c\}$.

28. $3[(x - \overline{x - 3}) + (2x + \overline{3 - 2x})]$.

CHAPTER IV.

MULTIPLICATION AND DIVISION.

MULTIPLICATION.

105. In algebra, as in arithmetic, **multiplication** is the process of taking one number as many times as there are units in another number. The result obtained is called the **product.**

The number taken or multiplied is called the **multiplicand,** and the number denoting how many times the multiplicand is taken is called the **multiplier.** The multiplicand and multiplier are **factors** of the product.

In multiplication the product is formed from the multiplicand as the multiplier is formed from unity.

106. It has been shown in § 75 that $a \times b$ or $-a \times (-b) = +ab$, and that $a \times (-b)$ or $-a \times b = -ab$; and hence, when two factors have *like* signs, their product is *positive;* and, when two factors have *unlike* signs, their product is *negative;* or, more briefly stated, *like signs give* +, *and unlike signs* −.

It follows that the product of several negative factors is negative when the number of factors is *odd* (1, 3, 5, 7, etc.), and positive when the number of factors is *even* (2, 4, 6, etc.).

Thus, $-a \times (-b) \times (-c) = ab \times (-c) = -abc$;
and $-a \times (-b) \times (-c) \times (-d) = ab \times cd = abcd$;
or $-a \times (-b) \times (-c) \times (-d) = -abc \times (-d) = abcd$.

107. Since $a^3 = a \times a \times a$ and $a^2 = a \times a$ (§ 28),

$$a^3 \times a^2 = a \times a \times a \times a \times a = a^5;$$

and generally since $a^m \times a^n = a^{m+n}$, it follows that *the exponent of any letter in a product is equal to the sum of its exponents in the several factors.*

Thus, $2a^5 \times 3a^3 \times a^2 = 6a^{5+3+2} = 6a^{10}$; $a^3b^4 \times a^2b^3 \times ab = a^6b^8$.

Monomials by Monomials.

108. **1.** What is the product of $5a^2b^3c$ and $3a^3bc^3$? Of $5a^2b^3c$ and $-3a^3bc^2$.

(1) $5a^2b^3c \times 3a^3bc^2 = 15a^5b^3c^3$

(2) $5a^2b^3c \times -3a^3bc^2 = -15a^5b^3c^3$

Since the monomials in (1) have like signs, their product is positive; and since in (2) they have unlike signs, their product is negative (§ 106). $5 \times 3 = 15$; $a^2 \times a^3 = a^5$; $b^3 \times b = b^3$; $c \times c^2 = c^3$. The product in (1) is $15a^5b^3c^3$, and in (2) $-15a^5b^3c^3$.

In multiplying two monomials, first determine the *sign* of the product, and, if $-$, write it; then find the product of the coefficients, and, if not 1, write it; and lastly annex each letter with an exponent equal to the sum of its exponents in the factors.

Multiply

2. $3a^2xy$ by $5abx$.

3. $12a^2bxy^2$ by $-3b^2cx^3y^2$.

4. $-2bx^3$ by $-4axy^2$.

5. $4a^2by^3$ by $-2a^3bcx$.

6. $-7ab^2c^3x^4$ by $2ac^2x^3$.

7. $9a^2b^4x^2y$ by $-10a^3bxy^3$.

8. $-4abcx^4$ by $-7a^3bc^4xy^4$.

9. $-8a^4c^2y^2z^3$ by $-5b^2c^4y^2z$.

10. $-c^3y^2z^3$ by $-11ac^2x^4y^3z^3$.

11. $-7a^2b^2c^2d^3$ by $a^3b^4c^5$.

12. $a^3m^5x^2y^3$ by $-b^2n^3xz^7$.

13. $20p^5q^4r^5t^2$ by $-\frac{1}{2}pqrs^4t$.

14. $-9abc^3xy^4$ by $-15a^3b^3x^4y^{10}$.

15. $-12b^3c^2y^4z^3$ by $\frac{3}{4}bc^5yz^5$.

16. $-3x^2y$ by $-2xy$ by $-3xy^2$.

17. $-4a^3x$ by $\frac{1}{2}ax^2$ by $-a^2x^3$.

109. To multiply monomials,

Multiply the numerical coefficients, and prefix the proper sign; then annex all the letters, giving to each an exponent equal to the sum of its exponents in the several factors.

Since a change in the order of the factors does not change the product (Commutative Law, § 80), for convenience arrange the literal factors in alphabetical order.

If a letter occurs in only one of the factors, it will have the same exponent in the product.

Polynomials by Monomials.

110. Since $3(4+6-2) = 3\times 4 + 3\times 6 - 3\times 2 = 24$, and generally $a(b+c-d) = ab + ac - ad$ (Distributive Law, § 83), it follows that *a polynomial may be multiplied by a monomial by multiplying each term of the polynomial by the monomial.*

1. Multiply $3a^2 - 5ab + 6abx$ by $-4a^2x$.

Process.

$$\begin{array}{l} 3a^2 - 5ab + 6abx \\ -\ 4a^2x \\ \hline -12a^4x + 20a^3bx - 24a^3bx^2 \end{array}$$

Write the two factors as at the left, and multiply each term of the multiplicand by $-4a^2x$.

Multiply

2. $a - b - c$ by $-5ac$; by $3ac$.
3. $x + y - z$ by $-x$; by $3x$; by $5xy$.
4. $a^2 - 2ab + b^2$ by $4ab$; by $-5ab^2$; by $3a^2b$.
5. $3x^3 - 2x^2y + 5xy^2 + y^3$ by $-3x^2y$; by $4xy^2$.
6. $3ax + 2cx^2 - 5x^3$ by $5cx^3$; by $-c^2x$.
7. $a^2b - 6a^3x - bx^3 + x^4$ by $-5ab^2x$; by $4a^2x$.
8. $ax^4 - a^2x^3 - 3a^3x + cy^2$ by $-axy^2$; by $5a^2x^2y^2$.
9. $a^2b^2x^3 - 7b^3y^3 - x^4$ by $-3abx$; by $6ab^2x^2$.
10. $12y^2 - 4xy + 2x^2$ by $-3x^2y$; by $\frac{1}{2}xy^2$.
11. $ab + \frac{1}{2}bc - cd$ by $\frac{3}{5}ac$; by $-\frac{2}{3}ad$.
12. $\frac{1}{2}a^2b^2 + \frac{3}{4}a^2b - \frac{1}{2}ab$ by $\frac{2}{3}ab$; by $-\frac{3}{4}a^2b^2$.
13. $xy^2 - x^2y^3 + \frac{2}{3}x^3y^4 - y^5$ by $-\frac{1}{2}x^2y$; by $\frac{2}{3}xy^2$.

111. To multiply a polynomial by a monomial,

Multiply successively each term of the polynomial by the monomial factor, connecting the several products with the proper signs.

Polynomials by Polynomials.

112. When the multiplier in arithmetic is expressed by two or more orders, as 45, the product is found by multiplying successively the multiplicand by the number denoted by each figure in the multiplier, and adding the partial products. Thus, $64 \times 45 = 64 \times 5 + 64 \times 40 = 320 + 2560 = 2880$.

In like manner in algebra, when the multiplier is a polynomial, the product is found by multiplying the multiplicand by each term of the multiplier, and adding the partial products, as shown below.

1. Multiply $a^2 - 2ab + b^2$ by $a + 3b$.

Process.

$$\begin{array}{l} a^2 - 2ab + b^2 \\ a + 3b \\ \hline a^3 - 2a^2b + ab^2 \\ \quad 3a^2b - 6ab^2 + 3b^3 \\ \hline a^3 + a^2b - 5ab^2 + 3b^3 \end{array}$$

Write the factors as at the left. Multiply first each term of $a^2 - 2ab + b^2$ by a, giving $a^3 - 2a^2b + ab^2$ as the first partial product; next multiply by $+3b$, giving $3a^2b - 6ab^2 + 3b^3$ as the second partial product, and write the like terms in the same column; lastly, add the two partial products.

2. Multiply $5 - 3x + 7x^2 + 8x^3$ by $10 - 8x + x^2$.

$$\begin{array}{l} 5 - 3x + 7x^2 + 8x^3 \\ 10 - 8x + x^2 \\ \hline 50 - 30x + 70x^2 + 80x^3 \\ \quad - 40x + 24x^2 - 56x^3 - 64x^4 \\ \qquad\qquad 5x^2 - 3x^3 + 7x^4 + 8x^5 \\ \hline 50 - 70x + 99x^2 + 21x^3 - 57x^4 + 8x^5 \end{array}$$

113. In multiplying polynomials, it is convenient to arrange the terms of both factors in such order that the like terms of the partial products shall fall in the same column. This is done by selecting a letter that occurs in several terms of each factor, and arranging the terms in both factors according to the powers of such letter. The letter thus selected is usually the *leading* letter in the two factors.

When the powers of the selected letter increase from left to right (as in 2 above), the polynomials are said to be arranged according to the *ascending* powers of the leading letter; and, when the powers of such letter decrease from left to right (as the powers of a in 1 above), the polynomials are said to be arranged according to the *descending* powers of the leading letter. The polynomial $a^3 - 2a^2bx + 3ab^2x^2 + b^3x^3$ is arranged according to the descending powers of a and the ascending powers of b and x.

3. Multiply $3a^2 + 2a - 5a^4b^2 + 4a^3b$ by $a^2 - 3a^3b + 2a$.

$2a + 3a^2 + 4a^3b - 5a^4b^2$
$2a + a^2 - 3a^3b$

Arrange both factors according to the ascending powers of a, as at the left.

4. Multiply $3 - 2x^2 + 6x + x^3$ by $4x^2 - x + 5$.

$x^3 - 2x^2 + 6x + 3$
$4x^2 - x + 5$

Arrange the multiplicand according to the descending powers of x, as at the left.

Multiply

5. $x^2 + 3xy$ by $x - 2y$.
6. $x^2 + 2x - 3$ by $2x - 5$.
7. $x + 4y$ by $x + 3y$.
8. $5m^2 + 2n - 1$ by $m^2 - 3n$.
9. $a^3 + ab^2$ by $a^3 - ab^2$.
10. $a^2 - 2ab + b^2$ by $a - b$.
11. $x - 5a$ by $ax - 3a^2$.
12. $x^2 - y^2$ by $x^2 + y^2$.
13. $8x^2 + 3ay$ by $8x^2 - 3ay$.
14. $x^2 + 2xy + y^2$ by $x - y$.
15. $m^2n - mn^2 - 3$ by $mn^2 - m^2n$.
16. $x^2 - 3x + 4$ by $x^2 - 6x$.
17. $x^2 - 2xy + y^2$ by $x^2 + 2xy + y^2$.
18. $2x^3 - 4x^2y + 5xy^2 - y^3$ by $x^2 - 3xy - 4y^2$.
19. $5x^4a - 3x^3a^2 + x^2a^3 - xa^4$ by $5x^2a^3 - 3xa^4$.
20. $a^3 + b^3 - 2ab + a^2 - b^3$ by $a^2 - ab$.
21. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 + 2xy + y^2$.
22. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.
23. $(a^2 + a - 20)(a - 4)$ by $a^3 - 5a + 6$.
24. $(y^2 - 2y^2 + z^2)(y - z)$ by $(y + z)(y - z)$.
25. $x^3 - 3x^2 + 5$ by $x^3 - x^2 - 3$.

114. To multiply polynomials,

Arrange the terms of each polynomial according to the ascending or descending powers of the same letter.

Multiply all the terms of the multiplicand by each term of the multiplier, and add the several partial products.

NOTE. In multiplying, observe carefully the laws of the signs.

EXERCISES.

Multiply

1. $x^2 - x + 1$ by $x + 1$.
2. $1 + x + x^2 + x^3$ by $1 - x$.
3. $x^2 + 2xy + y^2$ by $x + y$.
4. $a^2 - 2ab + b^2$ by $a - b$.
5. $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$.
6. $a^2 - 2ab + 4b^2$ by $a^2 + 2ab + 4b^2$.
7. $5x^2 - 3xy + y^2$ by $3x^2 + 5xy - y^2$.
8. $1 - a + a^2 - a^3$ by $1 + a - a^2$.
9. $5x^3 + 4x^2 + 3x + 2$ by $5x^3 - 4x^2$.
10. $x^3 - x^2y + xy^2 - y^3$ by $x + y$.
11. $a^6 + a^4b^2 + a^2b^4 + b^6$ by $a^2 - b^2$.
12. $3a^2 + 2b^2 - 5ab$ by $a^2 - 7ab + 1$.
13. $a^5 - a^4b + ab^4 - b^5$ by $a + b - 1$.
14. $x^3 + 3ax^2 + 3a^2x + a^3$ by $x^2 + 2ax + a^2$.
15. $x^4 - x^3 + x^2 - x + 1$ by $x^3 + x^2 - x - 1$.
16. $5x^2 - 7x + 1$ by $7x^2 + 2x + 4$.
17. $x^2 + 3x + 9$ by $x^2 - 3x + 9$.
18. $11x + x^3 - 24 - 4x^2$ by $4x + x^2 + 5$.
19. $10 + 2x + 6x^2 + x^3$ by $2x - 6x^2 + x^3$.
20. $x^4 - x^2y^2 + y^4$ by $3y^4 + 3x^2y^2 + 3x^4$.
21. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
22. $a^2 - 2ab + b^2 + c^2$ by $a^2 + 2ab + b^2 - c^2$.

23. $x^5 + y^5 + xy^4 + x^2y^3 + x^3y^2 + x^4y$ by $x - y$.

24. $(x^2 + a^2)(x + a)$ by $x - a$.

25. $(b + 2c)(b - c)$ by $(b + c)(b - 2c)$.

Find the product of

26. $a + b$ and $a - b$; $x + y$ and $x - y$.

27. $2a + b$ and $2a - b$; $2x + 3y$ and $2x - 3y$.

28. $m + n$ and $m - n$; $2m + n$ and $2m - n$.

29. $x + 2$, $x - 2$, and $x - 3$.

30. $x^2 - x + 1$, $x + 2$, and $x - 1$.

31. $x + a$, $x - 2a$, $x + 4a$, and $x - a$.

32. $a^2 + 2ab + b^2$, $a + b$, and $a - b$.

33. $x + a$, $x - a$, and $x^2 - 2a$.

34. What is the square of $a + b$? Of $a - b$?

35. What is the third power of $x + y$? Of $x - y$?

36. Find all the powers of $2x + c$ to the third.

37. Find the fourth power of $a + b$; of $a - b$.

38. Find the square of $a^2 - 2ab + b^2$.

39. Find the square of $x + y + z$; of $x - y - z$.

40. What is the third power of $3a - 2c$? Of $3x + 2y$?

41. What is the fourth power of $2x - 3y$?

115. An algebraic expression *may be simplified* by performing all the multiplications and divisions indicated, removing the parentheses, if any, and then collecting the like terms.

Simplify

42. $3a(a + b + c) - 2a(b - c)$.

43. $(a^2 + 2ab + b^2)(a - b) - 3a(ab + b^2)$.

44. $(a + b)(b + c) - b \times [b - (a - c)]$.

45. $(a + b)^2 - (a + b)(a - b) - (2b^2 - ab)$.

46. $2(a^2 - ab - b^2) - (a + b)^2$.

47. $(a+1)^3-(a-1)^3$.

48. $(x^2-xy+y^2)^2-2xy(-x^2-y^2)$.

49. $(x^2+xy-y^2)^2-(x^2-xy-y^2)^2$.

50. Substitute $a+2$ for x in x^3-2x^2+x.

51 Substitute $a-1$ for x, and $1-a$ for y, in $x^2+2xy+y^2$.

DIVISION.

116. Division is the inverse of multiplication (§ 77). In multiplication two factors are given to find their product; in division a product and one of its factors are given to find the other factor. Hence

117. Division is the process of finding one of two factors when their product and the other factor are given.

The given product is the **dividend**, the given factor the **divisor**, and the factor sought the **quotient**.

Since $3\times4=12$, $12\div3=4$ and $12\div4=3$; and since $a^3\times a^2=a^5$, $a^5\div a^3=a^2$, and $a^5\div a^2=a^3$, it follows, that, *if a product be divided by one of its factors, the quotient will be the other factor.*

MONOMIALS BY MONOMIALS.

118. The process of dividing one monomial by another is easily learned if it be kept in mind that it is the *inverse* of the process of multiplying two monomials.

119. In multiplying monomials, *like signs give plus, and unlike signs give minus* (§ 106); and hence the same law of the signs holds true in dividing one monomial by another.

Thus, $ab\div b=+a$, and $-ab\div(-b)=+a$; and $ab\div(-b)=-a$, and $-ab\div b=-a$ (§ 77).

120. In multiplying monomials, the product of their coefficients is the coefficient of the product (§ 109); and hence, in dividing one monomial by another, *the coefficient of the dividend*

is divided by the coefficient of the divisor, and the result is the coefficient of the quotient. Thus, the numerical coefficient of $12\,a^5b^3 \div 3\,a^3b^2$ is $12 \div 3 = 4$.

121. In multiplying monomials, the exponents of each letter are added (§ 109); and hence, in dividing monomials, *the exponent of each letter in the divisor is subtracted from the exponent of the same letter in the dividend.* Thus, $a^5 \div a^3 = a^{5-3} = a^2$; and $12\,a^5b^3 \div 3\,a^3b^2 = 3\,a^{5-3}b^{3-2} = 3\,a^2b$.

1. $21\,a^5b^3$ is a product, and $7\,a^2b^2$ is one of its two factors. What is the other factor?

Process. $21\,a^5b^3 \div 7\,a^2b^2 = (21 \div 7)(a^5 \div a^2)(b^3 \div b^2) = 3\,a^3b$.

The process may be illustrated by resolving both monomials into their prime factors, and then omitting the common factors; thus,

$$21\,a^5b^3 = 7 \times 3 \times aaaaabbb, \text{ and } 7\,a^2b^2 = 7 \times aabb.$$

$$\therefore\ 21\,a^5b^3 \div 7\,a^2b^2 = \frac{7 \times 3 \times aaaaabbb}{7 \times aabb} = 3 \times aaab = 3\,a^3b.$$

122. In the division of monomials it is often convenient to write the dividend over the divisor in the form of a fraction, as below.

2. Divide $28\,x^5y^4z$ by $-4\,x^3y^2$.

Process. $\dfrac{28\,x^5y^4z}{-4\,x^3y^2} = -7\,x^2y^2z$.

Proof. $-4\,x^3y^2 \times -7\,x^2y^2z = 28\,x^5y^4z$.

Since $28 \div (-4) = -7$, $x^5 \div x^3 = x^2$, $y^4 \div y^2 = y^2$, and $z \div 1 = z$, then $28\,x^5y^4z \div (-4\,x^3y^2) = -7\,x^2y^2z$.

It is not necessary to divide z by 1; z may simply be written in the quotient.

3. Divide $-56\,a^6x^4y^2$ (1) by $14\,a^3x^2y^2$; (2) by $-14\,a^3x^2y^2$.

Process. (1) $\dfrac{-56\,a^6x^4y^2}{+14\,a^3x^2y^2} = -4\,a^3x^2y^0 = -4\,a^3x^2$.

(2) $\dfrac{-56\,a^6x^4y^2}{-14\,a^3x^2y^2} = 4\,a^3x^2y^0 = 4\,a^3x^2$.

123. It is noted that y^0 is omitted in the quotient in Example 3, and this may need explanation.

Since $\frac{y^2}{y^2} = y^0$ (§ 121), and $\frac{y^2}{y^2} = 1$, it follows that $y^0 = 1$; and, since 1 may be omitted without affecting a quotient, y^0 may also be omitted. *Any letter which would appear in a quotient with 0 for an exponent may be omitted, since it is equal to 1.*

4. Divide $42\,a^4x^3y^2$ by $-14\,a^2x^5y^2$.

Process. $\frac{42\,a^4x^3y^2}{-14\,a^2x^5y^2} = -3\,a^2x^{-2}y^0 = -3\,a^2x^{-2}$.

By the rule for exponents (§ 121), $\frac{x^3}{x^5} = x^{3-5} = x^{-2}$. For a full discussion of negative exponents, see § 434.

5. Divide $-36\,b^5c^3y^2$ by $-12\,b^3cy$; by $18\,bc^2y^2$.

Divide

6. $-63\,a^2x^2y^5$ by $-7\,x^2y^3$.
7. $25\,m^5n^4x^7$ by $5\,mn^3x^5$.
8. $24\,a^5bcx^2$ by $-12\,a^4bx^2$
9. $-56\,p^2qr$ by $-7\,p^2r$.
10. $-120\,a^4x^5z^6$ by $-3\,ax^3z^5$.
11. $182\,m^5ny^3$ by $26\,m^4ny$.
12. $284\,a^9b^6c^3$ by $-71\,a^5b^5c^3$.
13. $-1728\,x^2y^2z^2$ by $-144\,xy^2z^2$.
14. $-512\,a^4x^3y^3z$ by $-64\,a^2y^2z$.
15. $343\,mn^3pr^3$ by $-49\,mn^2pr$.

124. To divide one monomial by another,

Divide the coefficient of the dividend by the coefficient of the divisor, prefixing the sign + if the signs are alike, and the sign − if the signs are unlike.

Annex to this coefficient each letter of the dividend, giving it an exponent equal to its exponent in the dividend less its exponent in the divisor.

Polynomials by Monomials.

125. Since division is the inverse of multiplication, a polynomial is divided by a monomial by dividing each term of the given polynomial by the monomial, observing the laws of the

signs. Thus, $(a+b-c)a=a^2+ab-ac$, and hence $(a^2+ab-ac) \div a = \frac{a^2}{a}+\frac{ab}{a}-\frac{ac}{a}=a+b-c$ (Distributive Law, § 83).

1. Divide $12\,a^3x^4-8\,a^2x^3y^2+4\,a^4x^2y^3$ by $4\,a^2x^2$.

Process. $\dfrac{12\,a^3x^4-8\,a^2x^3y^2+4\,a^4x^2y^3 \mid 4\,a^2x^2}{3\,ax^2-2\,xy^2+a^2y^3,\ \text{quotient.}}$

If preferred, the divisor may be written under the dividend and the quotient at the right, after the sign =. Thus,

$$\frac{12\,a^3x^4-8\,a^2x^3y^2+4\,a^4x^2y^3}{4\,a^2x^2}=3\,ax^2-2\,xy^2+a^2y^3$$

Divide

2. $3\,x^3y-3\,x^2y^2+3\,xy^3$ by $3\,xy$; by $-3\,xy$.
3. $4\,a^3b^2-8\,a^3b^3+12\,a^2b^3-4\,a^2b^2$ by $-4\,a^2b$.
4. $2\,ax^2y^2-4\,ax^3y^3-6\,axyz$ by $2\,axy$.
5. $3\,ax^3y-6\,a^2x^2y^2+6\,a^3xy^3-6\,a^4y^4$ by $-3\,ay$.
6. $8\,x^2y-16\,x^5y^2-4\,x^6$ by $4\,x^2$.
7. $12\,a^4x^6-4\,a^5x^5+20\,a^6x^4$ by $-2\,a^4x^3$.
8. $25\,x^2yz-20\,xy^2z+45\,xyz^2$ by $-5\,xyz$.
9. $60\,a^3b^3c^2-48\,a^2b^4c^2+36\,a^2b^2c^4$ by $12\,abc^2$.
10. $5\,x^3yz-14\,x^3y^3z^3-6\,x^5y^2z+20\,x^2y^2z^2$ by $-x^2yz$.

126. To divide a polynomial by a monomial,

Divide successively each term of the polynomial by the monomial.

Polynomials by Polynomials.

127. The first step in the division of one polynomial by another is to arrange the terms of each according to the ascending or descending powers of a common letter, if there be such, as illustrated below. This order of the terms is to be kept throughout the entire process.

1. Divide $4a^2b^4 + a^6 - b^6 - 4a^4b^2$ by $a^2 - b^2$.

Arrange the terms according to the descending powers of a.

$$\begin{array}{l|l} a^6 - 4a^4b^2 + 4a^2b^4 - b^6 & a^2 - b^2 \\ \underline{a^6 - a^4b^2} & \overline{a^4 - 3a^2b^2 + b^4} \\ -3a^4b^2 + 4a^2b^4 & \\ \underline{-3a^4b^2 + 3a^2b^4} & \\ a^2b^4 - b^6 & \\ \underline{a^2b^4 - b^6} & \end{array}$$

2. Divide $6x^2y^2 - 4xy^3 - 4x^3y + x^4 + y^4$ by $x^2 + y^2 - 2xy$.

Arrange the terms according to the descending powers of x.

$$\begin{array}{l|l} x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 & x^2 - 2xy + y^2 \\ \underline{x^4 - 2x^3y + x^2y^2} & \overline{x^2 - 2xy + y^2} \\ -2x^3y + 5x^2y^2 - 4xy^3 & \\ \underline{-2x^3y + 4x^2y^2 - 2xy^3} & \\ x^2y^2 - 2xy^3 + y^4 & \\ \underline{x^2y^2 - 2xy^3 + y^4} & \end{array}$$

3. Divide $24x^2y^2 - 32x^3y + 16x^4 - 8xy^3 + y^4$ by $2x - y$.

$$\begin{array}{l|l} 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4 & 2x - y \\ \underline{16x^4 - 8x^3y} & \overline{8x^3 - 12x^2y + 6xy^2 - y^3} \\ -24x^3y + 24x^2y^2 & \\ \underline{-24x^3y + 12x^2y^2} & \\ 12x^2y^2 - 8xy^3 & \\ \underline{12x^2y^2 - 6xy^3} & \\ -2xy^3 + y^4 & \\ \underline{-2xy^3 + y^4} & \end{array}$$

4. Divide $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.

$$\begin{array}{l|l} a^4 + a^2b^2 + b^4 & a^2 - ab + b^2 \\ \underline{a^4 - a^3b + a^2b^2} & \overline{a^2 + ab + b^2} \\ a^3b & \\ \underline{a^3b - a^2b^2 + ab^3} & \\ a^2b^2 - ab^3 + b^4 & \\ \underline{a^2b^2 - ab^3 + b^4} & \end{array}$$

It is observed that certain powers of a are wanting in the dividend. We may insert 0 for the second and third terms, or leave blank spaces for these terms, as at the left.

Divide

5. $6x^4 - 96$ by $3x - 6$.
6. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 - 3x^2y + 3xy^2 - y^3$.
7. $x^3 + 2x^2y + 2xy^2 + y^3$ by $x + y$.
8. $x^6 + x^5y + x^4y^2 + x^3y^4 + xy^5 + y^6$ by $x^3 + xy + y^3$.
9. $10a^4 - 27a^3b + 34a^2b^2 - 18ab^3 - 8b^4$ by $5a^2 - 6ab - 2b^2$.
10. $36a^4b^2 - 60a^3b^3 + 25a^2b^4$ by $6a^2b - 5ab^2$.
11. $x^2 + 2xy + y^2$ by $x + y$.
12. $x^2 - 2xy + y^2$ by $x - y$.
13. $9a^2 + 12ab + 4b^2$ by $3a + 2b$.
14. $25x^2 - 70xy + 49y^2$ by $5x - 7y$.
15. $16x^2 - 144$ by $4x - 12$.
16. $a^3 + b^3$ by $a + b$; $a^3 - b^3$ by $a - b$.
17. $x^3 + 1$ by $x + 1$; $1 - x^3$ by $1 - x$.
18. $27x^6 - 8y^3$ by $3x^2 - 2y$.
19. $125m^6 + 343$ by $5m^2 + 7$.
20. $m^4 - n^4$ by $m + n$.
21. $a^5 + b^5$ by $a + b$; $a^5 - b^5$ by $a - b$.
22. $x^6 - y^6$ by $x + y$; $x^6 - y^6$ by $x^3 - y^3$.
23. $125x^3 + 75x^2 + 15x + 1$ by $5x + 1$.
24. $216x^6 - 216ax^4 + 72a^2x^2 - 8a^3$ by $6x^2 - 2a$.
25. $x^4 - 17x^2 + 16$ by $x^2 + 5x + 4$.
26. $x^3 + a^3$ by $x^2 - ax + a^2$.
27. $27x^6 + 125$ by $3x^2 + 5$.
28. $a^7 + b^7$ by $a + b$; $a^6 + b^6$ by $a^2 + b^2$.
29. $x^9 - y^6$ by $x^3 - y^2$; $243x^5 - 1$ by $3x - 1$.
30. $1728m^6 - 343$ by $12m^2 - 7$.
31. $15x^4 - 14x^3 + 25x^2 - 6x - 9$ by $3x^2 - x - 1$.
32. $14x^4 - 10x^3 - 41x^2 + 25x + 15$ by $7x^2 - 5x - 3$.
33. $x^5 - 26x^3 + 6x^2 + 5x - 1$ by $x^3 - 5x^2 + 1$.

34. $1+y^6$ by $1-y^2+y^4$; $1-y^6$ by $1+y^2+y^4$.

35. x^8-y^8 by x^2+y^2; also by x^2-y^2.

36. $x^8+x^4y^4+y^8$ by $x^4+x^2y^2+y^4$.

37. a^9+1 by a^3+1.

38. $1-x-3x^2-x^5$ by $1+2x+x^2$.

39. $a^3x^6-b^6y^6$ by ax^2-b^2y.

40. $a^3+3a^2b+3ab^2+b^3$ by $a^2+2ab+b^2$.

41. $8x^3-36x^2+54x-27$ by $2x-3$.

42. $1+y^6-2y^3$ by y^2+1-2y.

43. $6x^4+4x^3y-9x^2y^2-3xy^3+2y^4$ by $2x^2+2xy-y^2$.

44. $24+38x+43x^2+34x^3+17x^4+6x^5$ by $6+5x+4x^2+3x^3$.

45. $x^7-571x^3+2118x^2+13824x+1360$ by $x^3+5x^2+36x+136$.

128. To divide one polynomial by another,

Arrange both dividend and divisor according to the powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient, and subtract the product from the dividend.

If there be a remainder, consider it as a new dividend, divide its first term by the first term of the divisor, and proceed as before.

FRACTIONAL COEFFICIENTS.

129. An algebraic number is integral if it has no denominator that contains a letter; and hence integral numbers may have *fractional coefficients.* Thus, $a^2-\frac{2}{3}ab+\frac{1}{4}b^2$ is an integral trinomial with fractional coefficients.

130. The fundamental processes with fractional coefficients are the same as those with integral coefficients.

1. Add $\frac{1}{2}ab^2 + \frac{1}{3}b^2c + \frac{2}{5}$ and $\frac{1}{4}ab^2 - \frac{1}{6}b^2c - \frac{1}{5}$.

2. Add $\frac{1}{2}a^2 + \frac{1}{4}ab - \frac{1}{3}b^2$, $\frac{3}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$, $\frac{1}{4}a^2 - ab + \frac{2}{3}b^2$, and $\frac{1}{2}a^2 - ab + \frac{3}{4}b^2$.

3. From $\frac{1}{3}x - \frac{5}{2}y - \frac{3}{4}xy$ take $\frac{1}{8}x - 2xy + \frac{3}{4}y$.

4. From $\frac{2}{3}a^2 - \frac{1}{4}ab + \frac{1}{2}b^2$ take $\frac{1}{2}a^2 - \frac{1}{3}ab - \frac{1}{2}b^2$.

5. Multiply $\frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{4}$ by $\frac{1}{2}x - \frac{1}{3}$.

6. Multiply $\frac{1}{2}a^2 - 2a + \frac{3}{2}$ by $\frac{1}{2}a + \frac{1}{3}$.

7. Divide $\frac{1}{6}a^3 - \frac{5}{36}a^2b + \frac{1}{6}ab^2 + \frac{2}{9}b^3$ by $\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{2}{3}b^2$.

8. Divide $\frac{1}{27}x^3 - \frac{1}{12}x^2 + \frac{1}{16}x - \frac{1}{64}$ by $\frac{1}{3}x - \frac{1}{4}$.

9. Multiply $\frac{1}{3}x^2 - \frac{2}{3}x - \frac{3}{4}$ by $\frac{1}{2}x - \frac{1}{4}$.

10. Multiply $\frac{2}{3}a^2 + ab + \frac{2}{3}b^2$ by $\frac{1}{3}a - \frac{1}{2}b$.

11. Divide $x^5 - \frac{1}{3}x^4 - \frac{1}{4}x^3 - \frac{1}{4}x^2 + \frac{1}{27}$ by $x^2 - \frac{1}{2}x - \frac{1}{3}$.

12. Divide $\frac{1}{4}x^3 + \frac{1}{72}x - \frac{1}{12}$ by $\frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{4}$.

13. Divide $\frac{1}{6}a^3 - \frac{5}{36}a^2b + \frac{1}{6}ab^2 + \frac{2}{9}b^3$ by $\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{2}{3}b^2$.

14. Divide $\frac{1}{4}x^4 - \frac{43}{36}x^2 + \frac{9}{16}$ by $\frac{1}{2}x^2 + \frac{3}{2}x - \frac{3}{4}$.

15. Divide $\frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3$ by $\frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{4}y^2$.

DETACHED COEFFICIENTS.

131. When two polynomials can be arranged according to the ascending or descending powers of a common letter, their product or quotient can be readily found by operating upon the coefficients detached for the purpose, and then supplying in the result the proper powers of the letters, as shown below.

1. Multiply $a^3 + 2a^2 + 3a + 1$ by $a^2 - 2a + 1$.

Detached coefficients,
$$\begin{array}{r} 1 + 2 + 3 + 1 \\ 1 - 2 + 1 \\ \hline 1 + 2 + 3 + 1 \\ -2 - 4 - 6 - 2 \\ \hline 1 + 2 + 3 + 1 \\ \hline 1 + 0 + 0 - 3 + 1 + 1 \end{array}$$

Supplying the powers of a, $a^5 + 0\,a^4 + 0\,a^3 - 3\,a^2 + a + 1$.

The product is $a^5 - 3\,a^2 + a + 1$.

2. Multiply $a^3 + 3ab^2 - 2b^3$ by $a^3 - 4a^2b + 3b^3$.

$$\begin{array}{r} 1+0+3-2 \\ \underline{1-4+0+3} \\ 1+0+3-2 \\ -4-0-12+8 \\ \underline{\qquad\qquad 3+0+9-6} \\ 1-4+3-11+8+9-6 \end{array}$$

Product, $a^6 - 4a^5b + 3a^4b^2 - 11a^3b^3 + 8a^2b^4 + 9ab^5 - 6b^6$.

3. Multiply $x^2 + \frac{1}{2}x + \frac{1}{8}$ by $x^2 - \frac{1}{2}x - \frac{1}{8}$.

$$\begin{array}{r} 1+\frac{1}{2}+\frac{1}{8} \\ \underline{1-\frac{1}{2}-\frac{1}{8}} \\ 1+\frac{1}{2}+\frac{1}{8} \\ -\frac{1}{2}-\frac{1}{4}-\frac{1}{16} \\ \underline{\qquad -\frac{1}{8}-\frac{1}{16}-\frac{1}{64}} \\ 1+0-\frac{1}{4}-\frac{1}{8}-\frac{1}{64} \end{array}$$

Product, $x^4 - \frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{64}$.

4. Multiply $a + bx + cx^2 + dx^3$ by $m - nx - rx^2$.

$$\begin{array}{r} a+\ b+\ c+\ d \\ \underline{m-\ n-\ r} \\ am+bm+cm+dm \\ -an-bn-cn-dn \\ \underline{\qquad -ar-br-cr-dr} \\ am+(bm-an)+(cm-bn-ar)+(dm-cn-br)-(dn+cr)-dr \end{array}$$

Product,

$$am+(bm-an)x+(cm-bn-ar)x^2+(dm-cn-br)x^3-(dn+cr)x^4-drx^5.$$

5. Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$.

6. Multiply $1 - 2x + 3x^2 + x^3$ by $2 - x + 2x^2$.

7. Multiply $2x^3 - 6x^2 + 5x - 2$ by $3x^2 - 4x + 5$.

8. Multiply $x^4 - x^2y^2 + y^4$ by $x^4 + x^2y^2 + y^4$.

9. Multiply $a^3 - 3a^2 + 2a - 1$ by itself.

10. Multiply $a^3 - 4a^2 + 11a - 24$ by $a^2 + 4a + 5$.

11. Multiply $4a^3 + 3a^2 + 2a + 1$ by $2a^3 - 3a^2 - 3$.

12. Multiply $\frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{2}{3}x + \frac{1}{3}$ by $2x^2 - \frac{1}{2}x - \frac{1}{4}$.

13. Multiply $ax^4 + bx^3 + cx + d$ by $mx^3 - nx - r$.

14. Find the square of $x^3 - 3x^2 - 6x + 2$.

15. Find the cube of $x^2 - 2x + 1$.

16. Divide $x^4 - 2x^3 + 8x - 3$ by $x^2 + 2x - 1$.

$$
\begin{array}{l|l}
x^4 + 0 \quad -2x^3 + 8x - 3 & x^2 + 2x - 1 \\
\underline{x^4 + 2x^3 \quad - x^2} & \underline{\qquad\qquad} \\
\quad -2x^3 \quad - x^2 + 8x & x^2 - 2x + 3 \\
\quad \underline{-2x^3 - 4x^2 + 2x} & \\
\qquad\qquad 3x^2 + 6x - 3 & \\
\qquad\qquad \underline{3x^2 + 6x - 3} &
\end{array}
$$

Process by Detached Coefficients.

$$
\begin{array}{l|l}
1 + 0 - 2 + 8 - 3 & 1 + 2 - 1 \\
\underline{1 + 2 - 1} & \underline{\qquad\qquad} \\
\quad -2 - 1 + 8 & 1 - 2 + 3 \\
\quad \underline{-2 - 4 + 2} & \\
\qquad\qquad 3 + 6 - 3 & \\
\qquad\qquad \underline{3 + 6 - 3} &
\end{array}
$$

$x^2 - 2x + 3$, quotient.

17. Divide $2x^4 + x^3 - 8x^2 + 17x - 12$ by $2x^2 - 3x + 4$.

$$
\begin{array}{l|l}
2 + 1 - \ 8 + 17 - 12 & 2 - 3 + 4 \\
\underline{2 - 3 + \ 4} & \underline{\qquad\qquad} \\
\quad 4 - 12 + 17 & 1 + 2 - 3 \\
\quad \underline{4 - \ 6 + \ 8} & \\
\qquad -\ 6 + \ 9 - 12 & \\
\qquad \underline{-\ 6 + \ 9 - 12} &
\end{array}
$$

$x^2 + 2x - 3$, quotient.

18. Divide $x^5 + 6x^3 - 4x^2 + 24x - 27$ by $x^3 + 2x - 3$.

$$
\begin{array}{l|l}
1 + 0 + 0 + 6 - 4 + 24 - 27 & 1 + 0 + 2 - 3 \\
\underline{1 + 0 + 2 - 3} & \underline{\qquad\qquad} \\
\quad 0 - 2 + 9 - 4 + 24 & 1 + 0 - 2 + 9 \\
\quad \underline{-2 - 0 - 4 + \ 6} & \\
\qquad\qquad 9 + 0 + 18 - 27 & \\
\qquad\qquad \underline{9 + 0 + 18 - 27} &
\end{array}
$$

$x^3 - 2x + 9$, quotient.

19. Divide $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.

20. Divide $4x^5 - x^3 + 4x$ by $2x^3 - 3x^2 + 2x$.

SYNTHETIC DIVISION.

132. An examination of the process of dividing one polynomial by another shows that it may be abridged by observing the following facts:

I. Only the *first* term of the divisor is used as a divisor in obtaining the successive terms of the quotient.

II. The products found by multiplying the first term of the divisor by the successive terms of the quotient are the same as the first terms of the corresponding dividends, *and hence these products may be omitted in the process.*

III. Only the first term of each successive remainder is divided to obtain the corresponding term of the quotient, *and hence the other terms of the several remainders need not be found.*

IV. The signs of the terms of the several partial products are changed in subtracting them, and this result may be attained *by changing the signs of all the terms of the divisor before multiplying them;* but, since the first term of each partial product is omitted (see II. above), it is only necessary *to change the signs of the remaining terms of the divisor.*

133. The manner in which these facts may be applied in abridging the division process is shown below. For convenience the divisor is written at the left in a vertical column, and the quotient below the dividend and partial products.

1. Divide $a^4 - 25a^2 + 40a - 16$ by $a^2 + 5a - 4$.

Full Process.

$$
\begin{array}{r|l}
 & a^4 + 0 - 25a^2 + 40a - 16 \\
 & a^4 + 5a^3 - 4a^2 \\
\hline
+5a & -5a^3 - 21a^2 + 40a \\
 & -5a^3 - 25a^2 + 20a \\
\hline
-4 & 4a^2 + 20a - 16 \\
 & 4a^2 + 20a - 16 \\
\hline
 & a^2 - 5a + 4, \text{ quotient.}
\end{array}
$$

Abridged Process.

$$\begin{array}{r|l|l} a^2 & a^4+0\quad -25a^2\quad +40a-16 & \\ \hline -5a & \quad -5a^3+\ 4a^2 & \\ +4 & \qquad\qquad 25a^2 & -20a \\ & & -20a+16 \\ \hline & a^2-5a\ +\ 4 & +\ 0\ +\ 0 \end{array}$$

The quotient is a^2-5a+4.

Since the first term of the divisor is used only as a *divisor*, its sign is not changed, and it is separated from the succeeding terms, *whose signs are changed*, by a horizontal line.

The first term of the quotient is $a^4 \div a^2 = a^2$, written below.

The first partial product, omitting a^4, is $(-5a+4)\times a^2 = -5a^3+4a^2$.

The second term of the quotient is $-5a^3 \div a^2 = -5a$, written below.

The second partial product is $(-5a+4)\times(-5a)=25a^2-20a$.

The sum of the terms in the third column is $+25a^2+4a^2-25a^2=+4a^2$, and the third term of the quotient is $+4a^2 \div a^2 = +4$, written below.

The third partial product is $(-5a+4)\times 4 = -20a+16$.

The sum of the fourth and fifth columns being each 0, the division is exact, and the quotient is a^2-5a+4. The number of 0's is one less than the number of terms in the divisor.

It is well to indicate where the division ends by a vertical line, as above.

Process by Detached Coefficients.

$$\begin{array}{r|l|l} 1 & 1+0-25\quad +40-16 & \\ \hline -5 & \quad -5+\ 4 & \\ +4 & \qquad\quad 25 & -20 \\ & & -20+16 \\ \hline & 1-5+\ 4 & +\ 0+\ 0 \end{array}$$

Or (2)

$$\begin{array}{r|l} 1 & 1+0-25+40-16 \\ \hline -5 & \quad -5+25-20 \\ +4 & \qquad\quad +\ 4-20+16 \\ \hline & -5+\ 4+\ 0+\ 0 \\ & 1-5+\ 4 \end{array}$$

Supplying the powers of a, the quotient is a^2-5a+4.

In the second process (2) the partial products are written diagonally downward to the right, each term being opposite

the term of the divisor multiplied. If preferred, this form may be used.

In this abridged process the several partial products or dividends are formed by addition or synthesis, and hence the process is called **synthetic division.**

It will be shown later (§ 215) that this method is of great advantage in factoring polynomials that cannot otherwise be readily factored. It may also be used with advantage in finding the integral roots of certain higher equations (§ 472).

2. Divide $2x^4 + x^3 - 22x^2 + 34x - 15$ by $x^2 + 3x - 5$.

$$\begin{array}{r|l} 1 & 2 + 1 - 22 \quad + 34 - 15 \\ -3 & \quad - 6 + 10 \\ +5 & \qquad\quad + 15 \;\big|\; - 25 \\ & \qquad\qquad\quad - \;\; 9 + 15 \\ \hline & 2 - 5 + \;\; 3 \;\big|\; + \;\; 0 + \;\; 0 \end{array}$$

The quotient is $2x^2 - 5x + 3$.

3. Divide $2x^5 + 3x^4 - 8x^3 - 20x^2 + 30x + 60$ by $2x^3 - 5x^2 + 10$.

$$\begin{array}{r|l} 2 & 2 + 3 - \;\; 8 \quad - 20 + 30 + 60 \\ +\;\; 5 & \quad + 5 + \;\; 0 \quad - 10 \\ +\;\; 0 & \qquad + 20 \;\big|\; + \;\; 0 - 40 \\ -\;\; 10 & \qquad\qquad\quad 30 + \;\; 0 - 60 \\ \hline & 1 + 4 + \;\; 6 \;\big|\; + \;\; 0 - 10 - \;\; 0 \end{array}$$

The quotient is $x^2 + 4x - 6$, with $-10x$ remainder.

Divide

4. $x^4 - x^3 - 7x^2 + x + 6$ by $x^2 - x - 6$.

5. $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.

6. $x^4 + 2x^2 - x + 2$ by $x^2 - x + 1$.

7. $1 + 2x^2 - x^3 + 2x^4$ by $1 - x + x^2$.

8. $x^6 - 4x^5 + 3x^4 + 2x^3 + 4x^2 - 15$ by $x^3 - 3x^2 + 5$.

9. $x^5 - 26x^3 + 6x^2 + 5x - 1$ by $x^3 - 5x^2 + 1$.

10. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 - 3x^2y + 3xy^2 - y^3$.

11. $x^4 - \frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{8}$ by $x^2 + \frac{1}{2}x + \frac{1}{8}$.

CHAPTER V.

SIMPLE EQUATIONS.

134. An **equation** is the expression of the equality of two numbers (§ 6).

An algebraic equation is written by means of symbols; but it may be read, or expressed orally, by words.

135. The number that precedes the sign of equality is called the **first member** of the equation, or the *left side;* and the number that follows the sign is called the **second member,** or the *right side.*

136. An equation that is true only for certain *particular values* of the letters therein is called an **equation of condition.** Thus, $3x + 3 = 15$, in which x equals 4 only, is an equation of condition.

An equation which is true for *all values* of the letters therein is called an **identical equation,** or, briefly, an **identity.** Thus the equation $a(a - b) = a^2 - ab$ is an identity, since it is true for all values of a and b. If, for example, $a = 5$, and $b = 4$, then $5(5 - 4) = 25 - 20$; that is, $5 = 5$.

In an identical equation the sign $\equiv$, called the **sign of identity,** may be used instead of the sign $=$. The identity $a + a \equiv 2a$ is read, "a plus a is identical with $2a$."

137. An equation which, when cleared of fractions, contains only the first power of the unknown number, is called an **equation of the first degree.** Thus, $ax - x = ab$, and $a + \frac{b}{x} = \frac{a}{b}$, are equations of the first degree.

The equation $\frac{x}{a} = \frac{a-b}{x}$ is not of the first degree, for, when cleared of fractions, it becomes $x^2 = a^2 - ab$, an equation of the second degree.

An equation of the first degree is also called a **simple equation.**

138. The **solution** of an equation is the finding of the value of the unknown number in it; i.e., the finding of a number which, substituted for the unknown number, will satisfy the equation. The value of the unknown number is called the *root* of the equation.

139. The process of solving an equation involves the making of such needed changes in the two members as *do not affect their equality.*

140. These changes are based on the following self-evident truths or principles, called

AXIOMS.

1. *If equals be added to equals, the sums will be equal.*
2. *If equals be subtracted from equals, the remainders will be equal.*
3. *If equals be multiplied by equals, the products will be equal.*
4. *If equals be divided by equals, the quotients will be equal.*
5. *Like powers of equals are equal.*
6. *Like roots of equals are equal.*
7. Generally, *if the same changes be made in equals, the results will be equal.*

141. These axioms, when applied to algebraic equations, show that

I. *The same number may be added to or subtracted from both members of an equation.* Thus, if $x = 12$, $x + 4 = 12 + 4$, and $x - 4 = 12 - 4$.

II. *Both members of an equation may be multiplied or divided by the same number.* Thus, if $x = 12$, $x \times 3 = 12 \times 3$; and if $x = 12$, $x \div 3 = 12 \div 3$.

III. *Like powers and like roots of both members of an equation are equal.* Thus, if $x = 4$, $x^2 = 16$; if $x^2 = 16$, $\sqrt{x^2} = \sqrt{16}$.

SOLUTION OF EQUATIONS.

142. In solving an equation, it is advantageous to transfer the terms that contain the unknown number to the first member, and the terms that contain only known numbers to the second member. This is called **transposing the terms.**

1. What is the value of x in the equation $x + 4 = 16$? In $x + b = a$?

(1) Since $x + 4 = 16$, $x + 4 - 4 = 16 - 4$ (Ax. 2). $\therefore x = 16 - 4 = 12$.

(2) Since $x + b = a$, $x + b - b = a - b$. $\therefore x = a - b$.

It is seen that $+4$ in (1), and $+b$ in (2), may be transposed to the second member *if the sign* $+$ *be changed to* $-$.

2. What is the value of x in the equation $x - 4 = 12$? In $x - b = a$?

(1) Since $x - 4 = 12$, $x - 4 + 4 = 12 + 4$ (Ax. 1). $\therefore x = 12 + 4 = 16$.

(2) Since $x - b = a$, $x - b + b = a + b$. $\therefore x = a + b$.

It is seen that -4 in (1), and $-b$ in (2), may be transposed to the second member *if the sign* $-$ *be changed to* $+$.

143. In like manner it may be shown that any term may be transposed from one member of an equation to the other, *provided its sign be changed.* Thus, if $5x - 8 = 16 + 3x$, then $5x - 3x = 16 + 8$. $\therefore 2x = 24$, and $x = 12$.

144. The signs of all the terms of an equation may be changed, since in effect this is the same as multiplying both members by -1 (Ax. 3).

145. When the terms of an equation are transposed and the unknown terms combined in one term, the value of the unknown number is found *by dividing both members by its coefficient.* Thus, if

(1) $3x = 30$, $3x \div 3 = 30 \div 3$ (Ax. 4). $\therefore x = 10$.

(2) $ax = b$, $\frac{ax}{a} = \frac{b}{a}$. $\therefore x = \frac{b}{a}$.

(3) $(a-b)x = a+b$, $\frac{(a-b)x}{a-b} = \frac{a+b}{a-b}$. $\therefore x = \frac{a+b}{a-b}$.

146. When the coefficient of the unknown term is a fraction, the value of the unknown number is found *by multiplying both members of the equation by the denominator* (Ax. 3), *and dividing both resulting members by the numerator.* Thus, if

(1) $\frac{3}{4}x = 15$, $3x = 15 \times 4 = 60$, and $x = 60 \div 3 = 20$.

(2) $\frac{a}{b}x = 2a$, $ax = 2ab$, and $x = \frac{2ab}{a} = 2b$.

It is really unnecessary to refer to Axioms 3 and 4 in the above solutions: for if $3x = 30$, it is evident that $x = \frac{1}{3}$ of 30, or $30 \div 3 = 10$; and if $\frac{3}{4}x = 15$, it is evident that $3x = 15 \times 4 = 60$, and $x = 60 \div 3 = 20$. Indeed, the relations involved are as evident as the axioms cited; and, besides, the pupil has applied these elementary principles throughout his entire course in arithmetic.

The equations hitherto solved, and those to be solved in this chapter, are *integral;* that is, equations no one of whose terms is fractional (§ 129). The solution of fractional equations will be presented in Chapter X.

Transpose the terms and find the value of x in

3. $3x - 12 = 18$.
4. $6x - 5 = 3x + 10$.
5. $5x + 15 = x + 35$.
6. $52 - 3x = 73 - 10x$.
7. $12x - 5 = 9x + 13$.
8. $3x - 3 = x + 5$.
9. $3x - 8 = 16 - 5x$.
10. $12x - 64 = 4(x - 5)$.
11. $2(3x - 4) = 3x + 13$.
12. $5x - (2x + 15) = 27 - 4x$.
13. $2x + 3 = 16 - (2x - 3)$.
14. $5x - 12 = 38 - (2x - 13)$.

Find the value of x in

15. $8x-(x-12)=3x+16$.
16. $4x-25=30-(2x-5)$.
17. $5x-6(x-5)=7(x-3)-5$.
18. $x-[3+(x-\overline{3+x})+2]=7$.
19. $6x-(12-x)=24-(3x-4)$.
20. $2x-(4x-1)=5x-(x+1)$.
21. $5x-3(x-1)=32-(5+2x)$.
22. $5x+5+6(x+2)=9(x+3)$.
23. $3x-(3x-\overline{3+2x})=2(3-\frac{1}{2}x)$.
24. $(x-2)(x+3)=(x-5)(x+1)+24$.
25. $(x-3)(x+1)=(x-1)(x-3)$.
26. $7x-(2x+10)=40+4(x-12)$.
27. $x+2(x-5)=5(13-x)-3$.
28. $2(x+3)-5(x-7)=3(x-14)-1$.
29. $32(x-5)=27(x-3)-4$.
30. $2(x+3)=5(x-1)-(-2x-1)$.
31. $5(x+1)-3(x+5)=2(6-x)+2$.
32. $2x+(a-b)=a+b$.
33. $ax+b=2a+b$.
34. $ax+b=a+bx$.
35. $3ax+b=2ax-(a-b)$.
36. $4x-2a=6a+2x$.
37. $ax-(a-b)=2a+b$.
38. $ax-ab=b^2-bx$.
39. $(ax+2)(a+1)=a(x+2)+5$.
40. $(a+x)(a-x)+b=(2+x)(1-x)$.
41. $ax-bx=(a+b)(a-b)$.
42. $a(x-a)=b(x-b)$.
43. $2(x-b)+3(x-2b)=2b$.
44. $(a+b)x+(a-b)x=2a^2b$.

147. To solve an integral equation of the first degree with one unknown number,

Transpose all the terms containing the unknown number to the first member, and all the other terms to the second member.

Combine the like terms in each member, and then divide both members by the coefficient of the unknown number.

To verify the result, *substitute the value of the unknown number in the original equation.*

PROBLEMS.

148. A **problem** is a question proposed for solution.

149. The solution of a problem consists of three processes or steps: namely,

I. The framing of algebraic expressions to denote the different numbers in the problem, called the **notation.**

II. The expression of the given relations between the known and unknown numbers in the form of an equation, called the **statement.**

III. The finding of the value of the unknown number, called the **solution of the equation.**

The first two of the above processes are usually called the statement.

It will be seen later that the statement of a problem may involve the forming of two or more related equations, and its solution the finding of the values of the unknown numbers in the several equations.

150. Any given equation may be regarded as the algebraic statement of a problem; and the invention of problems for which a given equation is a statement, is an interesting and useful introduction to the inverse process. Thus, $x + 2x = 45$ may be regarded as the algebraic statement of the problem, "A's age is twice B's age, and the sum of their ages is 45 years." The pupil has now had sufficient practice in the

notation and statement of problems to make such inventions of problems easy.

Invent problems of which the following equations are statements:

1. $x+3x=48$.
2. $5x-x=24$.
3. $3x+x=16$.
4. $2x-x=18$.
5. $x+x+8=24$.
6. $x+2x+8=44$.
7. $x+24=5x$.
8. $3x+4=x+20$.
9. $3x-2x=3$.
10. $2x+3x=400$.

11. Five times a certain number less 9 equals 15 less 3 times the number. What is the number?

Let $x =$ the number;
then $5x-9=15-3x$.
Transposing terms, $5x+3x=15+9$;
combining terms, $8x=24$;
dividing by 8, $x=3$.

VERIFICATION. $5\times 3-9=15-3\times 3$;
that is, $15-9=15-9$.

12. If 3 times a number less 6 be subtracted from 5 times the number, the difference will equal 24 less 4 times the number. What is the number?

Let $x =$ the number;
then $5x-(3x-6)=24-4x$.
Removing parenthesis, $5x-3x+6=24-4x$;
transposing terms, $5x-3x+4x=24-6$;
combining terms, $6x=18$;
dividing by 6, $x=3$.

VERIFICATION. $5\times 3-(3\times 3-6)=24-4\times 3$;
that is, $15-9+6=24-12$, or $12=12$.

13. Divide a line that is 25 inches long into two parts such that the greater shall be 4 inches longer than twice the less.

Let	x = length of less part;
then	$2x + 4$ = length of longer part.
Hence	$x + 2x + 4 = 25$.
Transposing terms,	$x + 2x = 25 - 4$;
combining terms,	$3x = 21$;
dividing by 3,	$x = 7$, less part;
	$2 \times 7 + 4 = 18$, longer part.

14. A father's age is twice his son's age, and 10 years ago his age was 3 times his son's age. What is the present age of each?

Let	x = son's present age;
then	$2x$ = father's present age;
	$2x - 10$ = father's age 10 years ago;
	$x - 10$ = son's age 10 years ago.
Hence	$2x - 10 = 3(x - 10) = 3x - 30$.
Transposing terms,	$2x - 3x = -30 + 10$;
combining terms,	$-x = -20$;
multiplying by -1,	$x = 20$, son's age;
	$2x = 40$, father's age.

15. The difference between two numbers is 17, and their sum is 93. What are the numbers?

16. What number added to 7 gives a sum equal to twice that number increased by 1?

17. The sum of two numbers is 54, and their difference 10. What are the numbers?

18. A man is 5 years younger than his brother, and the sum of their ages is 55 years. How old is each?

19. A father is 25 years old, and his son is 5 years old. In how many years will the father's age be twice the son's age?

20. A father's age is 4 times the son's age, and the difference of their ages is 27 years. What is the age of each?

21. Divide \$ 1000 among A, B, and C, so that A shall receive \$ 72 more than B, and C \$ 100 more than A.

22. A man sold a horse and buggy for $200; and one third the price of the horse was equal to one half the price of the buggy. Find the price of each.

SUGGESTION. Let $3x$ = price of horse, and $2x$ = price of buggy.

23. Divide the number 72 into two parts such that 3 times the greater may be equal to 5 times the less.

24. In a school of 143 pupils 5 times the number of boys equals 6 times the number of girls. How many of each?

25. I bought equal numbers of one-cent, two-cent, and four-cent stamps, paying $1.05 for all. How many did I obtain of each kind?

26. Divide the number 40 into three parts such that the second may be 3 times the first, and the third double the second.

27. At an election the number of votes cast for both candidates was 2560, and the successful candidate had a majority of 500 votes. How many votes did each receive?

28. Divide $2147 between A and B so that 8 times A's share may be equal to 11 times B's share.

29. A farmer, being asked how many sheep he had, replied that, if he had 12 sheep more, he would have 100 less than double the number he had. How many had he?

30. After A had received $12 from B, he had $19 more than B, and between them they had $75. How much had each at first?

31. The sum of $82.50 was paid in dollars, half-dollars, dimes, and five-cent pieces, an equal number of each piece being used. How many pieces of each kind?

32. A man paid $1000 for a certain number of horses at $60 each, 3 times as many cows at $30 each, and 20 times as many sheep at $5 each. How many of each did he buy?

33. A bill of $34 is paid in half-dollars and dimes, just 100 coins being used. How many of each were used?

34. A man divided $9 among a number of children, giving to some a quarter each, and to twice as many a dime each. How many children received the money?

35. The sum of $15,000 was raised among A, B, and C; B contributed $500 more than A; and C as much as A and B together. How much did each contribute?

36. Each of five brothers is 3 years older than his next younger brother, and the oldest is twice as old as the youngest. What is the age of each?

37. Divide the number 18 into two parts such that 5 times the greater increased by 4 shall be equal to 9 times the less diminished by 4.

38. Divide $4400 among A, B, and C, so that A may receive twice B's share, and B three times C's.

39. A company of 90 persons consists of men, women, and children. There are 4 more men than women, and 10 more children than both men and women. How many of each in the company?

40. A boy engaged to carry 100 glass vessels to a certain place on the condition that he should receive 3 cents for each one carried safely, and pay 9 cents for each one he broke. On settlement he received $2.40. How many did he break?

41. The sum of $5000 was divided among four persons, so that the first and second received together $2800; the first and third together, $2600; and the first and fourth together, $2200. How much did each receive?

42. After 34 gallons had been drawn from one of two equal casks, and 80 gallons from the other, one cask contained 3 times as much as the other. How much did each contain at first?

43. A brother is twice as old as his sister, and 3 years ago he was 3 times as old as she. What is the age of each?

44. In a brigade of 4500 men, the cavalry was 50 less than twice the number of artillery, and the infantry was 200 more than 8 times the cavalry. How many were there of each arm?

45. The distance between two cities is 1083 miles. From each a train sets out towards the other at the same hour, one at the rate of 35 miles an hour, and the other at the rate of 22 miles an hour. In how many hours will they meet?

46. Divide the number 70 into two parts such that the first plus 10 will be equal to the second multiplied by 3.

47. A and B engage in trade with equal capital. A gains $1600, and B loses $1900, and A's capital is now 8 times B's. What was the capital of each at first?

48. A boy has 5 more marbles in his right pocket than in his left; but, if he transfers 8 marbles from his left pocket to his right, he will then have 4 times as many marbles in his right pocket as in his left. How many marbles had he in each pocket at first?

49. A father gives to his four sons $2000, which they are so to divide that each elder son shall receive $50 more than his brother next younger. What is the share of each?

50. A newsboy has $6.50 in quarters, dimes, and five-cent pieces; and he has 3 times as many dimes as quarters, and 5 times as many five-cent pieces as dimes. How many pieces of each kind has he?

51. If a steamer sails 9 miles an hour downstream, and 5 miles an hour upstream, how far can it go downstream and return again in 14 hours?

52. If a steamer sails downstream at the rate of a mile in 5 minutes, and upstream at the rate of a mile in 7 minutes, how far downstream can it sail and return again in 1 hour?

CHAPTER VI.

FORMULAS.

SPECIAL FORMS IN MULTIPLICATION AND DIVISION.

151. An algebraic equation may be the expression of a general principle or a rule.

Thus, the equation $\frac{a+b}{2}+\frac{a-b}{2}=a$ is the algebraic expression of the principle, *the half of the sum of two numbers added to half their difference equals the greater number.*

152. The algebraic expression of a general principle or rule is called a **formula.**

153. The following formulas found by multiplication or division are of special utility in abridging algebraic processes, and also as a basis of factoring.

154. I. The square of the sum of two numbers.

$(a+b)^2=(a+b)(a+b)$; and it is found by multiplying, that $(a+b)(a+b)=a^2+2ab+b^2$. Hence

$$(a+b)^2\equiv a^2+2ab+b^2. \qquad (1)$$

Since a and b in (1) represent *any* two numbers, it follows that

The square of the sum of two numbers is the square of the first, plus twice the product of the first multiplied by the second, plus the square of the second.

Write by Formula (1) the square of

1. $x+y$.	4. $x+3$.	7. $4x+y$.
2. $a+x$.	5. $5+x$.	8. $2ax+3$.
3. $x+b$.	6. $3a+b$.	9. $3a+5b$.

155. II. The square of the difference of two numbers.

$(a-b)^2=(a-b)(a-b)$, and by multiplying it is found that $(a-b)(a-b)=a^2-2ab+b^2$. Hence

$$(a-b)^2\equiv a^2-2ab+b^2. \qquad (2)$$

Since a and b represent any two numbers, it follows from (2) that

The square of the difference of two numbers is the square of the first, minus twice the product of the first multiplied by the second, plus the square of the second.

Write by Formula (2) the square of

10. $x-y$.	13. $5-x$.	16. $x-5y$.
11. $x-a$.	14. a^2-b^2.	17. $ax-4$.
12. $x-3$.	15. $3a-b$.	18. $4a-3b$.

156. The foregoing principles are best fixed in the memory by means of their formulas, as follows:

$$(a+b)^2\equiv a^2+2ab+b^2. \qquad (1)$$

$$(a-b)^2\equiv a^2-2ab+b^2. \qquad (2)$$

157. These two formulas may be united in one by the use of the double sign $\pm$, read "plus or minus," thus:

$$(a\pm b)^2\equiv a^2\pm 2ab+b^2. \qquad (3)$$

If the upper sign $(+)$ or the lower sign $(-)$ is used in the first member of the formula, the same sign must be used in the second member.

158. If both terms of a binomial have the same sign ($a+b$ or $-a-b$), all the terms of its square will be positive; but, if the two terms of a binomial have unlike signs ($a-b$ or $-a+b$), the second term of its square will be negative.

Thus, $(a+b)^2$ or $(-a-b)^2=a^2+2ab+b^2$; and $(a-b)^2$ or $(-a+b)^2=a^2-2ab+b^2$.

Write the square of

19. $a + 3x$.
20. $x - 3y$.
21. $2m + 1$.
22. $1 - 2m$.
23. $a^2 - 2b$.
24. $2a - 3b$.
25. $3a^2 - 2b^2$.
26. $-ax + 2y$.
27. $-ab - 2c^3$.
28. $a^3b - 2cx$.
29. $x^4y + 25$.
30. $4ab - 2xy$.

159. III. The product of the sum and the difference of two numbers.

It is found by multiplying, that

$$(a + b)(a - b) \equiv a^2 - b^2. \qquad (4)$$

Hence

The product of the sum and the difference of two numbers is the difference of their squares.

Write the product of

31. $(x + y)(x - y)$.
32. $(x + 4)(x - 4)$.
33. $(3x + 3)(3x - 3)$.
34. $(2a + 3b)(2a - 3b)$.
35. $(ax - b)(ax + b)$.
36. $(1 + 3x^2y)(1 - 3x^2y)$.
37. $(4a^2x - 3y)(4a^2x + 3y)$.
38. $(5a^2b^2 + 1)(5a^2b^2 - 1)$.

160. IV. The square of any polynomial.

The square of a trinomial may be readily found by Formula (1) or (2) by including two of its terms in a parenthesis, thus changing it to a binomial.

Thus, since $a + b + c = (a + b) + c$ (Associative Law, § 82),

$$\begin{aligned}(a + b + c)^2 &= [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2 \\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\end{aligned}$$

39. Find the square of $a - b - c$.

$$\begin{aligned}(a - b - c)^2 &= [(a - b) - c]^2 = (a - b)^2 - 2(a - b)c + c^2 \\ &= a^2 - 2ab + b^2 - 2ac + 2bc + c^2 \\ &= a^2 + b^2 + c^2 - 2ab - 2ac + 2bc.\end{aligned}$$

40. Find the square of $a + b + c - d$.

Since $a + b + c - d = (a + b) + (c - d)$,

$$[(a + b) + (c - d)]^2 = (a + b)^2 + 2(a + b)(c - d) + (c - d)^2$$
$$= a^2 + 2ab + b^2 + 2ac + 2bc - 2ad - 2bd + c^2 - 2cd + d^2$$
$$= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2ad + 2bc - 2bd - 2cd.$$

161. An inspection of the foregoing equations shows that the square of a polynomial having three terms or four terms is made up of (1) *the sum of the squares of each of its terms,* and (2) *twice the product of each term multiplied in succession by each of the terms that follow it.*

It may be shown by multiplying, that the square of any given polynomial is *the sum of the squares of its several terms and twice the product of each term multiplied in succession by each of the terms that follow it.*

Find the square of

41. $x + y - z$.

42. $x - y + z$.

43. $x + y - 1$.

44. $x - y - z$.

45. $2a - b + c$.

46. $a + 3b - c$.

47. $x + y + z + v$.

48. $x - y - z - v$.

49. $a - 2b - c + 2d$.

50. $a + b - c - d$.

51. $a^2 + b^2 - c^2 - d^2$.

52. $a^2 + b^2 - 2c + d$.

162. V. The product of two binomials with a common term.

It may be found by multiplying, that

(1) $(x+a)(x+b) = x^2+ax+bx+ab \equiv x^2+(a+b)x+ab$;
(2) $(x+a)(x-b) = x^2+ax-bx-ab \equiv x^2+(a-b)x-ab$;
(3) $(x-a)(x+b) = x^2-ax+bx-ab \equiv x^2+(-a+b)x-ab$;
(4) $(x-a)(x-b) = x^2-ax-bx+ab \equiv x^2+(-a-b)x+ab$.

163. It is seen from these formulas that the product of any two binomials with a common term is made up of (1) *the square of the common term,* (2) *the product of the common term multiplied by the algebraic sum of the two unlike terms,* and (3) *the product of the unlike terms.*

164. By the aid of the foregoing formulas, the product of any two binomial factors with a common term may be written without multiplying, care being taken to observe that the coefficient of the second term of the product is *the algebraic sum of the two unlike terms*, and that the last term of the product is *the product of the unlike terms*. Thus,
$(x+4)(x-7)=x^2-3x-28$, and $(x-4)(x+7)=x^2+3x-28$.

Find the product of

53. $(x+4)(x+3)$.
54. $(x-6)(x-4)$.
55. $(x+8)(x-3)$.
56. $(x-9)(x+7)$.
57. $(x+3)(x-6)$.
58. $(x+5)(x-13)$.
59. $(x-10)(x-6)$.
60. $(x-12)(x+9)$.
61. $(x-15)(x-9)$.
62. $(x-a)(x-b)$.
63. $(a-6)(a+11)$.
64. $(9-x)(9-y)$.
65. $(x-2a)(x-3)$.
66. $(x+5c)(x-3c)$.
67. $(x-4b)(x-4)$.
68. $(x-ab)(x+2ab)$.

165. VI. The product of any two binomials.

It is found by multiplying $x+a$ by $y+b$ that the product of *any* two binomials is made up of

I. *The product of the first terms of the binomials* (xy).

II. *The algebraic sum of the cross-products of the binomial terms;* i.e., the product of the first term of each binomial multiplied by the second term of the other $(bx+ay)$.

III. *The product of the second terms* (ab).

Thus, (1), $(2x+5)(x-7)=2x^2+[2x\times(-7)+x\times 5]-35$
$=2x^2-9x-35$.

(2), $(x-b)(3x+2b)=3x^2+(2bx-3bx)-2b^2$
$=3x^2-bx-2b^2$.

Find the product of

69. $(3x-5)(x-7)$.
70. $(2x-7)(3x+5)$.
71. $(4m-5)(3m-4)$.
72. $(3x-1)(x-3)$.
73. $(3x+1)(x-2)$.
74. $(x-7)(5x-12)$.
75. $(2x+y)(3x-2y)$.
76. $(2-3ab)(3-2ab)$.
77. $(2a-3b)(a+b)$.
78. $(5x-4y)(x+2y)$.
79. $(2m+3mn)(m-mn)$.
80. $(3a-2b)(a+b)$.
81. $(3x+b)(x-2b)$.
82. $(5y-a)(2y-2a)$.
83. $(2x-y)(2x+3y)$.
84. $(3a+5b)(5a-7b)$.

DIVISION BY BINOMIAL FACTORS.

166. Since division is the inverse of multiplication, it follows from the foregoing formulas that

(1) $\dfrac{a^2+2ab+b^2}{a+b}=a+b$; (3) $\dfrac{a^2+(b+c)a+bc}{a+b}=a+c$;

(2) $\dfrac{a^2-2ab+b^2}{a-b}=a-b$; (4) $\dfrac{a^2-(b+c)a+bc}{a-b}=a-c$.

167. A trinomial that is the product of any two binomials may be divided by either binomial factor by inspection.

Thus, $\dfrac{a^2-3a-28}{a+4}=a-7$, and $\dfrac{a^2-3a-28}{a-7}=a+4$.

Divide by inspection

1. $x^2-2xy+y^2$ by $x-y$.
2. $a^4+2a^2b+b^2$ by a^2+b.
3. x^2-6x+9 by $x-3$.
4. $9a^2+6ab+b^2$ by $3a+b$.
5. $4a^2x^2+12ax+9$ by $2ax+3$.
6. $25x^2-10xy+y^2$ by $5x-y$.
7. $9a^2-12ab+4b^2$ by $3a-2b$.
8. $4x^4-12x^2y^2+9y^4$ by $2x^2-3y^2$.
9. $x^2+5x-24$ by $x-3$.
10. $a^2-16a+60$ by $a-6$.
11. $x^2-5ax+6a^2$ by $x-3a$.
12. $x^2+2cx-15c^2$ by $x+5c$.

168. There are several classes of binomials which are divisible by a binomial factor, and usually the quotient can be written directly. The following cases are often of special value in factoring.

169. I. Difference of two squares.

Since $(a+b)(a-b)=a^2-b^2$,

$$(1)\ \frac{a^2-b^2}{a+b}=a-b; \quad (2)\ \frac{a^2-b^2}{a-b}=a+b.$$

Hence the difference of the squares of two numbers is divisible *by their sum or by their difference.*

Divide by inspection

13. x^2-y^2 by $x-y$.

14. x^4-y^4 by x^2+y^2.

15. x^4-16 by x^2-4.

16. $9x^2-9$ by $3x+3$.

17. $4a^2-9b^2$ by $2a-3b$.

18. $9a^2x^2-b^2$ by $3ax-b$.

19. $100-9a^2b^2$ by $10-3ab$.

20. $1-9x^4y^2$ by $1-3x^2y$.

21. $25a^2x^4-1$ by $5ax^2+1$.

22. $16x^4y^2-9$ by $4x^2y-3$.

170. II. Sum and difference of two cubes.

It is found by dividing, that

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2; \qquad (1)$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2. \qquad (2)$$

171. Formula (1) shows that the sum of the cubes of two numbers is divisible by their sum, the quotient being *the square of the first number, minus the product of the first multiplied by the second, plus the square of the second.*

Formula (2) shows that the difference of the cubes of two numbers is divisible by their difference, the quotient being *the square of the first number, plus the product of the first multiplied by the second, plus the square of the second.*

Divide by inspection

23. x^3+y^3 by $x+y$.
24. x^3-y^3 by $x-y$.
25. $8a^3+b^3$ by $2a+b$.
26. $8a^3-b^3$ by $2a-b$.
27. x^3-27b^3 by $x-3b$.
28. $1-8y^3$ by $1-2y$.
29. $8y^3-1$ by $2y-1$.
30. $x^3y^3-z^3$ by $xy-z$.
31. $x^3y^3+z^3$ by $xy+z$.
32. $27+x^3y^3$ by $3+xy$.
33. x^6-y^6 by x^2-y^2.
34. x^6+y^6 by x^2+y^2.
35. $1-64x^3y^3$ by $1-4xy$.
36. $27x^3y^3-8$ by $3xy-2$.

172. III. Sum and difference of other like powers.

It is found by dividing, that

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3; \tag{1}$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3; \tag{2}$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4; \tag{3}$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4. \tag{4}$$

It is also found by division that a^7+b^7, a^9+b^9, and so on, are each divisible by $a+b$; and that a^7-b^7, a^9-b^9, and so on, are each divisible by $a-b$.

173. It will, however, be found by trial that a^2+b^2, a^4+b^4, a^6+b^6, a^8+b^8, and so on, are not divisible by $a+b$ or $a-b$, and also that a^3-b^3, a^5-b^5, and so on, are not divisible by $a+b$.

It can be proved that, if n is any positive integer,

I. a^n+b^n *is divisible by* $a+b$ *if* n *is odd, and by neither* $a+b$ *nor* $a-b$ *if* n *is even.*

II. a^n-b^n *is divisible by* $a-b$ *if* n *is odd, and by both* $a+b$ *and* $a-b$ *if* n *is even.*

174. It is seen from Formulas (2) and (4) in § 172 that the terms of the quotient are all positive when the divisor is $a-b$; and from (1) and (3), that the terms are alternately positive and negative when the divisor is $a+b$. It is also seen that the exponents of a in the quotient decrease, and those of b increase, from left to right.

Divide by inspection

37. x^4-y^4 by $x+y$.
38. x^4-y^4 by $x-y$.
39. x^5+y^5 by $x+y$.
40. x^5-y^5 by $x-y$.
41. x^6-y^6 by $x+y$.
42. x^6-y^6 by $x-y$.
43. x^4-1 by $x-1$.
44. $1-x^4$ by $1+x$.
45. $1+x^5$ by $1+x$.
46. $16-x^4$ by $2-x$.
47. x^6-64 by $x+2$.
48. x^4-81 by $x-3$.
49. $8x^3+1$ by $2x+1$.
50. x^3+27y^3 by $x+3y$.

Miscellaneous Exercises.

Write the square of

1. $x+y$.
2. $2x-y$.
3. $m+1$.
4. $1-m$.
5. $2-3a$.
6. $2x-3y$.
7. $3x^2-2y^2$.
8. a^2+2b.
9. $5-bc$.
10. $2ax+y^2$.
11. $ab-c$.
12. $7-2ax$.
13. $c+d+e$.
14. $a-b+c$.
15. $x+y-z$.
16. $1+c-d$.
17. $3a-b+1$.
18. $a-2b+1$.
19. $x-2y+3$.
20. $a+2b+3c$.
21. $5a-b-c$.
22. $m+n+r+s$.
23. $m-n+r-s$.
24. $a+b-c-1$.

Write the product of

25. $(x+y)(x-y)$.
26. $(m^2-n)(m^2+n)$.
27. $(2y^2+1)(2y^2-1)$.
28. $(1+3ax)(1-3ax)$.
29. $(5-3x)(5-3x)$.
30. $(7a+5)(7a+5)$.
31. $(4a^2+2b)(4a^2-2b)$.
32. $(x^2+y^2)(x^2-y^2)$.
33. $(x^3-y^3)(x^3+y^3)$.
34. $(a+7)(a-4)$.
35. $(x-3a)(x-4a)$.
36. $(x+b)(x-c)$.
37. $(x-m)(x+n)$.
38. $(m-5)(m+12)$.
39. $(m+r)(m+s)$.
40. $(x^2-x+1)(x+1)$.
41. $(x^2+x+1)(x-1)$.
42. $(x^3+x^2+x+1)(x-1)$.
43. $(a^2-ax+x^2)(a+x)$.
44. $(x^2+ax+a^2)(x-a)$.
45. $(x^2-2x+4)(x+2)$.
46. $(x^2+3x+9)(x-3)$.
47. $(x^3-x^2y+xy^2-y^3)(x+y)$.
48. $(x^3+x^2y+xy^2+y^3)(x-y)$.

Write the quotient of

49. $\dfrac{x^3+y^3}{x+y}$.
50. $\dfrac{a^3-x^3}{a-x}$.
51. $\dfrac{x^4-y^4}{x-y}$.
52. $\dfrac{a^4-x^4}{a+x}$.
53. $\dfrac{a^3-64}{a-4}$.
54. $\dfrac{a^4-81}{a+3}$.
55. $\dfrac{x^4-9}{x^2+3}$.
56. $\dfrac{1-a^4}{1+a^2}$.
57. $\dfrac{a^4-1}{a-1}$.
58. $\dfrac{x^5-y^5}{x-y}$.
59. $\dfrac{8x^3+y^3}{2x+y}$.
60. $\dfrac{8a^3-27b^3}{2a-3b}$.
61. $\dfrac{9a^2x^2-16b^4}{3ax+4b^2}$.
62. $\dfrac{4m^4n^2-25}{2m^2n+5}$.
63. $\dfrac{a^2-5a-24}{a-8}$.
64. $\dfrac{x^2+4x-32}{x+8}$.
65. $\dfrac{a^4-2a^2b^2+b^4}{a^2-b^2}$.
66. $\dfrac{x^4-2x^2y^4+y^8}{x^2-y^4}$.

CHAPTER VII.

FACTORING.

175. The **factors** of a number are the numbers which, multiplied together, will produce the number (§ 22).

Since $a \times 1 = a$, every number is equal to the product of itself and 1, and hence the number itself and 1 are factors of every number.

176. A number which has no integral factors except itself and 1 is a *prime number* (§ 22).

A number which has two or more integral factors besides itself and 1 is a *composite number*. Hence

177. Every composite number may be resolved into two or more integral factors besides itself and 1.

In giving the factors of a number, the number itself and 1 are usually omitted. A prime factor is a prime number.

178. **Factoring** is the process of resolving a composite number into its prime factors.

Skill in factoring is of great utility in abridging algebraic work, and the acquisition of such skill requires much practice.

CASE I.

179. **Monomials.**

Since $12 = 3 \times 2 \times 2$, and $a^3 = a \times a \times a$, the prime factors of $12\,a^3$ are 3, 2, 2, a, a, a.

In like manner any monomial may be resolved into its prime factors *by factoring the coefficient, and taking the base of each literal factor as many times as there are units in its exponent* (§ 29).

Resolve into prime factors

1. $6a^3b^2$.
2. $10a^2bx^2$.
3. $-9x^5y^2$.
4. $-15m^2nx^2$.
5. $49a^3cd^2$.
6. $-121x^2y^4$.
7. $35a^2b^2c^2$.
8. $-81a^3x^3$.
9. $100m^2n^3s$.

180. A number that is composed of two equal factors is a *perfect square*, and one of the two equal factors of such a number is its *square root*.

181. Since $(3a^3)^2 = 3a^3 \times 3a^3 = 3^2 \times a^{3\times 2} = 9a^6$,

$$\sqrt{9a^6} = \sqrt{9} \times \sqrt{a^6} = 3 \times a^{6\div 2} = 3a^3.$$

Hence

The square root of a monomial is found by *extracting the square root of the coefficient, and dividing the exponent of each letter by 2.*

10. Resolve $16a^4b^2$ into two equal factors.

$$\sqrt{16a^4b^2} = 4a^2b\ ;\ \therefore\ 16a^4b^2 = 4a^2b \times 4a^2b.$$

$\sqrt{16a^4b^2} = \pm 4a^2b$ (§ 335), but in this chapter only the *positive* square root is considered.

Resolve into two equal factors

11. $25a^6x^4$.
12. $81x^4y^8$.
13. $121m^4n^8r^6$.
14. $49m^2n^4s^{10}$.
15. $64a^2b^6c^4$.
16. $144x^2y^2z^4$.

182. In like manner the cube root of a monomial is found *by extracting the cube root of the coefficient, and dividing the exponent of each letter by 3.*

Thus, $\sqrt[3]{125a^6b^3x^9} = \sqrt[3]{125} \times a^{6\div 3}b^{3\div 3}x^{9\div 3} = 5a^2bx^3$.

Case II.

183. Polynomials whose terms have a common monomial factor.

This case presents no difficulty requiring explanation.

1. What are the factors of $3ax^2 + 6a^2by - 3a^2c$?

It is seen by inspection that $3a$ is a factor of each term, and hence, dividing by $3a$ and writing the quotient in a parenthesis with $3a$ as the other factor, we have

$$3ax^2 + 6a^2by - 3a^2c = 3a(x^2 + 2aby - a^2c).$$

2. Resolve into factors $x^3y + x^2y^2 + xy^3$.

$$x^3y + x^2y^2 + xy^3 = xy(x^2 + xy + y^2).$$

Resolve into factors

3. $10a^2b + 5ab^2$.
4. $14x^2yz - 63xy^2z$.
5. $ax^3 - a^2x^2y + axy^2$.
6. $a^4b - a^3b^2 + a^2b^3$.
7. $a^3x^2 + a^2y - a^2$.
8. $9a^2bx - 15a^2by - 3a^2b$.
9. $15x^3y - 10x^3y^2 + 5x^3z$.
10. $4x^4y - 12x^4y^2 - 16x^4y^3 + 8x^4y^4$.
11. $3x^3y^2 - 6x^4y^2 + 9x^2y^4 - 12x^3y^3$.
12. $ay^2 - 3a^2y^2z + 5a^3y^2z^2 - a^4y^4z^3$.

Case III.

184. Polynomials whose terms grouped have a common factor. A polynomial of four terms can sometimes be so arranged that the first two terms and the last two terms have a common *binomial* factor.

Thus, $ax + ay + bx + by = (ax + bx) + (ay + by) = (a + b)x + (a + b)y$; and, dividing $(a + b)x + (a + b)y$ by $a + b$, we have $x + y$; and hence $(a + b)x + (a + b)y = (a + b)(x + y)$.

1. Resolve into factors $ax^2 - ay + by - bx^2$.

$$ax^2 - ay + by - bx^2 = (ax^2 - bx^2) - (ay - by$$
$$= (a - b)x^2 - (a - b)y = (a - b)(x^2 - y).$$

The inclosing of the two terms, $-ay + by$, in a parenthesis preceded by the sign $-$, involves the changing of the signs (§ 104).

2. Resolve into factors $2ax - 4ay - 3bx + 6by$.

$$2ax - 4ay - 3bx + 6by = (2ax - 3bx) - (4ay - 6by)$$
$$= (2a - 3b)x - (2a - 3b)2y = (2a - 3b)(x - 2y).$$

Resolve into factors

3. $ab + by + ax + xy$.
4. $a^2 + ax - ay - xy$.
5. $ay - ab - bx + xy$.
6. $xy - 2my + 2mn - nx$.
7. $ax^2 - ay^2 + bx^2 - by^2$.
8. $y^3 + y^2 + y + 1$.
9. $a^3 - 3b - a^2b + 3a$.
10. $ax^2y + aby^2 - abx^2 - ay^3$.
11. $6a^3 + 4a^2 - 9a - 6$.
12. $6a^2x^2 - 4a^2y^2 - 3bx^2 + 2by^2$.

Case IV.

185. Trinomials which are perfect squares.

Since $(a+b)^2 = a^2 + 2ab + b^2$ (§ 154), $\sqrt{a^2 + 2ab + b^2} = a + b$;
and since $(a-b)^2 = a^2 - 2ab + b^2$ (§ 155), $\sqrt{a^2 - 2ab + b^2} = a - b$.

186. It is thus seen that a trinomial is a perfect square, if, when arranged according to the powers of some letter, its first and last terms are perfect squares and positive, and its second term is twice the product of their square roots. Thus, $9x^2 + 6xy + y^2$ is a perfect square. Hence

The square root of a trinomial which is a perfect square is found *by connecting the square roots of the terms which are squares with the sign of the remaining term.*

Thus, $\sqrt{9x^2 - 12xy + 4y^2} = \sqrt{9x^2} - \sqrt{4y^2} = 3x - 2y$.

187. A trinomial which is a perfect square may be resolved into two binomial factors *by extracting its square root, and making the result one of its two equal factors.*

1. Resolve into factors $4x^2 - 12xy^2 + 9y^4$.

$\sqrt{4x^2} = 2x$; $\sqrt{9y^4} = 3y^2$; $2x - 3y^2$ = the square root.
$\therefore 4x^2 - 12xy^2 + 9y^4 = (2x - 3y^2)(2x - 3y^2)$.

Resolve into factors

2. $4x^2 + 4xy + y^2$.
3. $a^2 + 6ab + 9b^2$.
4. $a^4 - 2a^2b^2 + b^4$.
5. $9x^2 - 6xy + y^2$.
6. $4x^4 - 20x^2y^2 + 25y^4$.
7. $x^2 - 2x + 1$.

8. $4y^2 - 4y + 1$.
9. $9x^2 - 24xy + 16y^2$.
10. $25x^2 - 10xy^2 + y^4$.
11. $144x^4 - 120x^2y + 25y^2$.
12. $1 - 10a^2y^2 + 25a^4y^4$.
13. $25 + 30ab + 9a^2b^2$.
14. $4 - 40ab + 100a^2b^2$.
15. $x^2y^4 - 6xy^2 + 9$.
16. $16 + 40ab^3c^3 + 25a^2b^4c^6$.
17. $121a^2 + 2200ab^4 + 10000b^8$.
18. $m^4n^2 - 40m^2ns^2 + 400s^4$.
19. $(a+b)^2 - 4(a+b) + 4$.
20. $(x-y)^2 - 6(x-y) + 9$.
21. $(m^2-n)^2 - 10(m^2-n)n + 25n^2$.
22. $a^2m + a^2n + b^2m + b^2n - 2abn - 2abm$.

Arranging terms,
$$\begin{aligned}&(a^2m - 2abm + b^2m) + (a^2n - 2abn + b^2n)\\ &= (a^2 - 2ab + b^2)m + (a^2 - 2ab + b^2)n\\ &= (a-b)^2m + (a-b)^2n = (a-b)^2(m+n)\\ &= (a-b)(a-b)(m+n).\end{aligned}$$

23. $a^2x^2 + 2a^2xy + a^2y^2 - b^2x^2 - 2b^2xy - b^2y^2$.

188. Since the square of a polynomial is made up of the sum of the squares of its several terms and twice the product of each term multiplied in succession by all the terms that follow it (§ 161), the square root of a polynomial which is a perfect square may be found *by connecting the square roots of the several terms which are the squares with the proper signs.*

189. The signs of the terms of the root are determined from the several terms of the polynomial which are the *products*, and this can usually be done with little difficulty by inspection.

Thus, $a^2 - 2bc + 2ab + c^2 - 2ac + b^2$
$= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc = (a + b - c)^2$.

It is seen that each product containing c, viz., $-2ac$, $-2bc$, is negative, and so it is inferred that c has the sign $-$.

Resolve into two equal factors

24. $x^2 + 2xy + 2xz + y^2 + 2yz + z^2$.
25. $x^2 - 2xy + z^2 - 2yz + 2xz + y^2$.
26. $a^4 - 4a^2b + 4a^2 + 4b^2 - 8b + 4$.

Case V.

190. Binomials expressing the difference of two squares.

Since $(a+b)(a-b)=a^2-b^2$ (§ 159),

$$a^2-b^2=(a+b)(a-b).$$

It is thus seen that the difference of two perfect squares is equal to the product of the sum and the difference of their square roots. Hence

191. A binomial expressing the difference of two perfect squares is resolved into two factors *by extracting the square root of each of its terms, and then taking the sum of their roots for one factor, and the difference of their roots for the other factor.*

1. Resolve into factors $9x^4y^2-4a^2b^2$.

$$\sqrt{9x^4y^2}=3x^2y;\ \sqrt{4a^2b^2}=2ab.$$
$$\therefore\ 9x^4y^2-4a^2b^2=(3x^2y+2ab)(3x^2y-2ab).$$

2. Resolve into factors $3ax^4-12ay^2$.

$$3ax^4-12ay^2=3a(x^4-4y^2);\ \sqrt{x^4}=x^2;\ \sqrt{4y^2}=2y.$$
$$\therefore\ 3ax^4-12ay^2=3a(x^2+2y)(x^2-2y).$$

3. Resolve into factors $(m+n)^2-(m-n)^2$.

$$\sqrt{(m+n)^2}=m+n;\ \sqrt{(m-n)^2}=m-n.$$
$$\therefore\ (m+n)^2-(m-n)^2=\overline{(m+n}+\overline{m-n)}\,\overline{(m+n}-\overline{m-n)}$$
$$=(m+n+m-n)(m+n-m+n)=2m\times 2n=4mn.$$

Resolve into factors

4. $9a^4b^2-16c^2$.
5. $x^2y^2-4y^2z^2$.
6. x^4-y^4.
7. $9y^2-1$.
8. $4a^2x^2-36a^2$.
9. $16a^4x^4-49z^2$.
10. $1-81z^4$.
11. y^4-16z^4.
12. $121-a^4x^8$.
13. $x^4-49y^4z^2$.
14. $144-25a^2y^2$.
15. $x^4-(a-b)^2$.
16. $a^2-(x-y)^2$.
17. $4x^4-(a+b)^2$.

18. $(x+y)^2-(x-y)^2$.
19. $(a+b)^2-(c+d)^2$.
20. $(m-n)^2-(m+n)^2$.
21. $(a-b)^2-(c-d)^2$.
22. $(a+b)^4-(a-b)^4$.
23. $(2a+3)^2-(3a-4)^2$.
24. $(3a+2b)^2-(2a-3b)^2$.
25. $(5x+2y)^2-(x-3y)^2$.

192. Some polynomials of four or more terms can be so written as to express the difference of two perfect squares, and their factors can thus be readily found.

26. Resolve into factors $x^2+2xy+y^2-z^2$.

$$x^2+2xy+y^2-z^2=(x^2+2xy+y^2)-z^2$$
$$=(x+y)^2-z^2=(x+y+z)(x+y-z).$$

27. Resolve into factors $4a^2-x^4+4x^2y-4y^2$.

$$4a^2-x^4+4x^2y-4y^2=4a^2-(x^4-4x^2y+4y^2)$$
$$=4a^2-(x^2-2y)^2=(2a+x^2-2y)(2a-x^2+2y).$$

Special care must be taken to change the signs of terms when put in a parenthesis preceded by the sign $-$, and also when a parenthesis preceded by the sign $-$ is removed. Thus, above, $-x^4+4x^2y-4y^2$ becomes $-(x^4-4x^2y+4y^2)$, and $-(x^2-2y)$ becomes $-x^2+2y$.

28. Resolve into factors $x^2+y^2-a^2-b^2-2xy+2ab$.

$$x^2+y^2-a^2-b^2-2xy+2ab=x^2-2xy+y^2-(a^2-2ab+b^2)$$
$$=(x-y)^2-(a-b)^2=(x-y+a-b)(x-y-a+b).$$

Resolve into factors

29. $4a^4-4a^2b-x^2y^2+b^2$.
30. $4x^4-9x^2+6x-1$.
31. $a^2b^2+c^2-2abc-25a^2b^2$.
32. $a^2-4x^2y^2-2ab+b^2$.
33. $1-2x+x^2-16a^2b^4$.
34. $a^2+4b^2-4ab-4x^2-9y^2-12xy$.
35. $a^2-b^2-c^2+2a+2bc+1$.
36. $x^2+y^2-2xy-2mn-m^2-n^2$.
37. $x^2+y^2-s^2-z^2+2xy+2sz$.
38. $a^2-2ac-b^2-d^2-2bd+c^2$.

193. A polynomial of the general form $a^4 + a^2b^2 + b^4$ may be written as the difference of two squares by adding to it a^2b^2, and then subtracting a^2b^2 (the number added) from the result.

Thus, $a^4 + a^2b^2 + b^4 = a^4 + a^2b^2 + b^4 + a^2b^2 - a^2b^2$

$= (a^4 + 2\,a^2b^2 + b^4) - a^2b^2$

$= (a^2 + b^2)^2 - a^2b^2 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$.

39. Resolve into factors $x^4 + x^2 + 1$.

$$x^4 + x^2 + 1 = x^4 + x^2 + 1 + x^2 - x^2$$
$$= (x^4 + 2\,x^2 + 1) - x^2 = (x^2 + 1)^2 - x^2$$
$$= (x^2 + x + 1)(x^2 - x + 1).$$

40. Resolve into factors $9\,a^4 - 3\,a^2b^2 + b^4$.

$$9\,a^4 - 3\,a^2b^2 + b^4 = 9\,a^4 - 3\,a^2b^2 + b^4 + 9\,a^2b^2 - 9\,a^2b^2$$
$$= 9\,a^4 + 6\,a^2b^2 + b^4 - 9\,a^2b^2 = (3\,a^2 + b^2)^2 - 9\,a^2b^2$$
$$= (3\,a^2 + 3\,ab + b^2)(3\,a^2 - 3\,ab + b^2).$$

Resolve into factors

41. $x^4 + x^2y^2 + y^4$.

42. $x^4 + 2\,x^2y^2 + 9\,y^4$.

43. $m^4 - 8\,m^2n^2 + 4\,n^4$.

44. $a^8 + a^4b^2 + b^4$.

45. $x^8 + 2\,x^4y^4 + 9\,y^8$.

46. $4\,x^4 - 16\,x^2y^2 + 9\,y^4$.

194. In like manner any polynomial that can be made a perfect square by adding to it a number, can be factored by adding such number, and subtracting it from the result, and then proceeding as in § 192.

Thus, $a^4 + 4\,b^4 = (a^4 + 4\,a^2b^2 + 4\,b^4) - 4\,a^2b^2 = (a^2 + 2\,b^2)^2 - 4\,a^2b^2$

$= (a^2 + 2\,b^2 + 2\,ab)(a^2 + 2\,b^2 - 2\,ab)$.

47. Resolve into factors $x^8 + 4\,y^4$.

$$x^8 + 4\,y^4 = x^8 + 4\,x^4y^2 + 4\,y^4 - 4\,x^4y^2$$
$$= (x^4 + 2\,y^2)^2 - 4\,x^4y^2 = (x^4 + 2\,y^2 + 2\,x^2y)(x^4 + 2\,y^2 - 2\,x^2y).$$

195. If the number added and subtracted be not a perfect square, the resulting factors will contain a radical, or indicated root (§ 370).

Thus, $a^2 + ab + b^2 = a^2 + ab + b^2 + ab - ab$

$= (a + b)^2 - ab = (a + b + \sqrt{ab})(a + b - \sqrt{ab})$.

Resolve into factors

48. $x^4 + 4$. 50. $x^2 + 4x$. 52. $a^4 + 3a^2b^2 + 4b^4$.

49. $64 + y^4$. 51. $x^4 + 2x^2y^2$. 53. $a^4 - 6a^2b^2 + b^4$.

Case VI.

196. Trinomials having binomial factors with a common term.

Since by § 162 $(x + a)(x + b) = x^2 + (a + b)x + ab$,

conversely $x^2 + (a + b)x + ab = (x + a)(x + b)$.

Hence a trinomial of the form $x^2 + mx + p$ may be resolved into two binomial factors, *if its third term is the product of two factors whose algebraic sum is the coefficient of the second term.*

197. There are two cases:

I. When the final term is *positive.*

II. When the final term is *negative.*

198. I. *When the final term is positive.*

By § 163, we have, conversely,

$$x^2 + (a + b)x + ab \equiv (x + a)(x + b);$$
$$x^2 - (a + b)x + ab \equiv (x - a)(x - b).$$

199. It is seen from these formulas, that, if the *third* term of the given trinomial is positive, the *second* terms of its binomial factors will have the *same sign* as the middle term of the trinomial; and (2) that the *sum* of the second terms of the binomial factors is the *coefficient* of the middle term of the trinomial.

1. Resolve into factors $x^2 + 7x + 12$.

$$12 = 3 \times 4;\ 3 + 4 = 7.$$
$$\therefore x^2 + 7x + 12 = (x + 3)(x + 4).$$

2. Resolve into factors $x^2 - 12x + 35$.

$$35 = -7 \times (-5);\ -7 + (-5) = -12.$$
$$\therefore x^2 - 12x + 35 = (x - 7)(x - 5).$$

Resolve into factors

3. $x^2+7x+10$.
4. $x^2-13x+22$.
5. $x^2-14x+45$.
6. $x^2+9x+20$.
7. $x^2+15x+56$.
8. $x^2-11x+28$.

200. II. *When the final term is negative.*

By § 162, conversely, we have

$$x^2+(a-b)x-ab=(x+a)(x-b);$$
$$x^2+(-a+b)x-ab=(x-a)(x+b).$$

201. It is seen from these equations that when the final term of a trinomial is *negative*, the second terms of its two binomial factors have *unlike signs*, and (2) that the *algebraic sum* of the second terms of the binomial factors is the *coefficient* of the middle term of the trinomial.

9. Resolve into factors $x^2+3x-28$.

$$-28=7\times(-4);\ 7-4=3.$$
$$\therefore x^2+3x-28=(x+7)(x-3).$$

The third term, -28, also equals -7×4; but $-7+4=-3$, whereas the coefficient of the second term of the trinomial is $+3$.

10. Resolve into factors $x^2-4x-45$.

$$-45=-9\times5;\ -9+5=-4.$$
$$\therefore x^2-4x-45=(x-9)(x+5).$$

It is seen from Examples 9 and 10 that the sign of the middle term of the trinomial is the sign of the numerically greater second term of the binomial factors.

Resolve into factors

11. $x^2+5x-24$.
12. $x^2-7x-60$.
13. $x^2-4x-45$.
14. $x^2+2x-63$.
15. $x^2+10x-56$.
16. $x^2-3x-70$.
17. x^2-x-72.
18. x^2+x-42.

202. It is thus seen that a trinomial whose binomial factors have only one common term may be resolved into its factors (1) *by taking the square root of the first term for the first or common term of the two factors,* and then (2) *finding for their second terms two numbers such that their product is the final term of the trinomial, and their algebraic sum the coefficient of its middle term.*

Resolve into factors

19. $x^2 + 8x + 16$.

20. $x^2 + 14x + 40$.

21. $x^2 + 16x + 63$.

22. $x^2 + 18x + 72$.

23. $y^2 + 20y + 96$.

24. $x^2 - 15ax + 54a^2$.

25. $x^2 - 8ax + 15a^2$.

26. $x^2 - 15x + 56$.

27. $x^2 - 17x + 52$.

28. $y^2 - 27y + 140$.

29. $x^2 + 3x - 28$.

30. $x^2 + 7x - 18$.

31. $x^2 + 5x - 66$.

32. $x^2 + x - 132$.

33. $y^2 + y - 182$.

34. $x^2y^2 - 9xy - 22$.

35. $x^2y^2 - 5xy - 104$.

36. $x^2 - 7ax - 60a^2$.

37. $x^2 - 2x - 99$.

38. $z^2 - z - 272$.

39. $x^2 + 32x + 175$.

40. $x^2 + 10x - 75$.

41. $x^2 - 40x + 400$.

42. $x^2 + x - 156$.

43. $y^2 + 17y + 66$.

44. $x^2 - 23x + 76$.

45. $x^2 - 15x - 54$.

46. $y^2 + 11y - 42$.

47. $y^2 - 7y - 170$.

48. $y^2 + 24y + 108$.

49. $x^4 + 17x^2 + 66$.

50. $x^6 - 15x^3 + 56$.

51. $x^8 + 2x^4 - 99$.

52. $y^6 + 7y^3 - 8$.

53. $y^{10} - 19y^5 + 48$.

54. $x^2 + (b + c)x + bc$.

55. $x^2 + (a - c)x - ac$.

56. $y^4 - (a^2 + b^2)y^2 + a^2b^2$.

203. The required factors of the final term can usually be found by inspection; but when the term is large, or contains many factors, it may be advantageous to resolve it into successive sets of factors, beginning with small prime divisors as first factors, and then taking their multiples, as is shown at the right.

Coeff.	33 (*Sum*)
$270 = 2 \times$	135
$3 \times$	90
$5 \times$	54
$6 \times$	45
$9 \times$	30
$10 \times$	27
$15 \times$	18

57. Resolve into factors $x^2 + 33x + 270$.

$270 = 15 \times 18$; $15 + 18 = 33$.

$\therefore\ x^2 + 33x + 270 = (x + 15)(x + 18)$.

Resolve into factors

58. $x^2 - x - 240$.
59. $x^2 + 31x + 240$.
60. $y^2 + 10y - 299$.
61. $y^2 - 14y - 480$.
62. $x^2 - 47x + 540$.
63. $x^2 - x - 420$.
64. $x^2 + x - 1260$.
65. $x^2 + 38x + 240$.
66. $x^2 + 7x - 1320$.
67. $x^2 + x - 552$.

Case VII.

204. Trinomials having binomial factors with unlike terms. We have considered the factoring of trinomials when their binomial factors have *like* terms, as $(a \pm b)^2$, and also when their binomial factors have a *common* term, as $(x \pm a)(x \pm b)$.

In the remaining case the binomial factors have *unlike terms*, and the trinomial has the form of $ax^2 + bx + c$.

205. It follows from § 165, that, when a trinomial is the product of two binomial factors with unlike terms,

I. The *first term* of the trinomial is the product of the *first terms* of its binomial factors.

II. The *third term* of the trinomial is the product of the *second terms* of its binomial factors.

III. If the third term of the trinomial is *positive*, the *second* terms of its binomial factors will both have *the sign of its middle term.*

IV. If the third term of the trinomial is *negative*, the *second* terms of its binomial factors will have *unlike signs.*

V. The middle term of the trinomial is the *algebraic sum* of the *cross-products* of the terms of its binomial factors.

1. Resolve into factors $6x^2 - 19x + 10$.

$6x^2 = 3x \times 2x$; $10 = -2 \times (-5)$; $3x \times (-2) + 2x(-5) = -19x$.

$\therefore 6x^2 - 19x + 10 = (3x - 2)(2x - 5)$.

Since the second term of the trinomial is negative, and the third term positive, the second term of each binomial factor is *negative.*

$6x^2 = 6 \times x^2$ or $6x \times x$ or $3x \times 2x$, and $10 = 2 \times 5$ or 1×10. It is seen that 1 and 10 are not the factors to be taken, and that $3x$ and $2x$ are the only factors of $6x^2$ which by cross-products with 2 and 5 will give $-19x$, the second term of the trinomial.

2. Resolve into factors $4x^2 + 4xy - 3y^2$.

$4x^2 = 2x \times 2x$; $-3y^2 = 3y \times (-y)$; $2x \times 3y + 2x \times (-y) = 4xy$.

$\therefore 4x^2 + 4xy - 3y^2 = (2x + 3y)(2x - y)$.

Since the algebraic sum of the cross-products must be *positive*, the greater product must be positive; and hence $3y$ is $+$, and y is $-$.

3. Resolve into factors $6x^2 - xy - 35y^2$.

$6x^2 = 3x \times 2x$; $-35y^2 = -5y \times 7y$; $3x \times (-5y) + 2x \times 7y = -xy$.

$\therefore 6x^2 - xy - 35y^2 = (2x - 5y)(3x + 7y)$.

Resolve into factors

4. $3x^2 - 22x + 35$.

5. $6x^2 - 11x - 35$.

6. $12m^2 - 31m + 20$.

7. $3x^2 - 10x + 3$.

8. $3x^2 - 5x - 2$.

9. $5x^2 - 47x + 84$.

10. $3x^2 - 5bx - 2b^2$.

11. $6x^2 - xy - 2y^2$.

12. $5x^2 + 6xy - 8y^2$.

13. $2m^2 + m^2n - 3m^2n^2$.

14. $3a^2 + ab - 2b^2$.

15. $10y^2 - 12ay + 2a^2$.

16. $4x^2 + 4xy - 3y^2$.

17. $15x^2 - 4xy - 35y^2$.

206. A trinomial of the form $ax^2 + bx + c$ can be readily factored by Case VI. *if it be first changed into an equivalent trinomial having unity for the coefficient of its first term.*

18. Resolve into factors $3x^2 - 31x + 56$.

Multiplying by 3, and dividing the result by 3, we have

$$3x^2 - 31x + 56 = \frac{(3x)^2 - 31(3x) + 168}{3},$$

$$= \frac{(3x - 24)(3x - 7)}{3} = (x - 8)(3x - 7).$$

It is seen that $3x$ is treated as x in Case VI.

19. Resolve into factors $12x^2 + 47x + 45$.

$$12x^2 + 47x + 45 = \frac{(12x)^2 + 47(12x) + 540}{12},$$

$$= \frac{(12x + 20)(12x + 27)}{12} = (3x + 5)(4x + 9).$$

Since $12 = 4 \times 3$, the first factor, $(12x + 20)$, is divided by 4; and the second factor, $(12x + 27)$, by 3.

Resolve into factors

20. $3x^2 + 14x + 15$.
21. $7x^2 + 20x + 12$.
22. $5x^2 - 17x + 12$.
23. $6x^2 + 5x - 6$.
24. $3x^2 - x - 10$.
25. $6x^2 - x - 40$.
26. $6x^2 - 13x + 6$.
27. $3x^2 + 5x - 12$.
28. $3x^2 - 17x + 20$.
29. $10x^2 + 23x - 21$.

GENERAL METHOD OF FACTORING TRINOMIALS.

NOTE. The study of this method may be omitted until the pupil reaches quadratic equations (Chapter XV.).

207. Any trinomial of the form $x^2 + bx + c$ may be factored by changing it to an equivalent expression of *the difference of two squares.*

Thus, whatever may be the value of b, $x^2 + bx$ is made a

perfect square by the addition of $\left(\frac{b}{2}\right)^2$, for $x^2+bx+\left(\frac{b}{2}\right)^2$ $=\left(x+\frac{b}{2}\right)^2$ (§ 186). Hence, by adding and subtracting $\left(\frac{b}{2}\right)^2$ to x^2+bx+c, we have

$$x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c=\left(x+\frac{b}{2}\right)^2-\frac{b^2}{4}+c$$

$$=\left(x+\frac{b}{2}\right)^2-\frac{b^2-4c}{4}=\left(x+\frac{b}{2}+\frac{\sqrt{b^2-4c}}{2}\right)\left(x+\frac{b}{2}-\frac{\sqrt{b^2-4c}}{2}\right).$$

It is evident that if b^2-4c is a perfect square, both factors will be rational.

1. Find the factors of x^2+6x+8.

$$x^2+6x+8=(x^2+6x+9)-9+8=(x+3)^2-1$$
$$=(x+3-1)(x+3+1)=(x+2)(x+4).$$

2. Find the factors of $x^2+9x+20$.

3. Find the factors of $3x^2-5x-2$.

$$3x^2-5x-2=3(x^2-\tfrac{5}{3}x-\tfrac{2}{3})=3[x^2-\tfrac{5}{3}x+(\tfrac{5}{6})^2-\tfrac{25}{36}-\tfrac{2}{3}]$$
$$=3[(x-\tfrac{5}{6})^2-\tfrac{49}{36}]=3(x-\tfrac{5}{6}+\tfrac{7}{6})(x-\tfrac{5}{6}-\tfrac{7}{6})=3(x+\tfrac{1}{3})(x-2).$$

4. Find the factors of $2x^2-3mx-2m^2$.

$$2x^2-3mx-2m^2=2\left(x^2-\frac{3m}{2}x-m^2\right)$$
$$=2\left[x^2-\frac{3m}{2}x+\left(\frac{3m}{4}\right)^2-\frac{9m^2}{16}-m^2\right]=2\left[\left(x-\frac{3m}{4}\right)^2-\frac{25m^2}{16}\right]$$
$$=2\left(x-\frac{3m}{4}+\frac{5m}{4}\right)\left(x-\frac{3m}{4}-\frac{5m}{4}\right)=2(x+\tfrac{1}{2}m)(x-2m).$$

208. In like manner any trinomial of the form ax^2+bx+c may be factored. For

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right]$$
$$=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$$
$$=a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right).$$

This is a general formula for all of the more important cases of factoring, and the preceding methods of factoring trinomials may be considered as special cases under it. It may be used with advantage in the solution of certain quadratic equations (§ 451).

209. Any trinomial may be directly factored by substituting for a, b, and c, in the formula in § 208, their values in the given trinomial.

Thus, in $3x^2 - 10x + 3$, $a = 3$, $b = -10$, and $c = 3$. Substituting these values in the formula, we have

$$x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = x + \frac{-10}{6} + \frac{\sqrt{100 - 36}}{6} = x - \frac{1}{3};$$

$$x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = x + \frac{-10}{6} - \frac{8}{6} = x - 3.$$

Hence $3x^2 - 10x + 3 = 3(x - \frac{1}{3})(x - 3)$.

Find by general formula the factors of

5. $3x^2 + 5x - 12$.
6. $x^2 + 7x + 12$.
7. $2x^2 + x - 28$.
8. $3x^2 - 13x + 14$.
9. $2x^2 - 5mx - 3m^2$.
10. $3x^2 - 5nx - 2n^2$.

SUM OR DIFFERENCE OF LIKE POWERS.

210. It follows from § 171, and also from Case V., that the *difference* of the *even* powers of two numbers can be resolved into two or more binomial factors. Thus,

$a^2 - b^2 \equiv (a + b)(a - b)$;

$a^4 - b^4 \equiv (a^2 + b^2)(a^2 - b^2) \equiv (a^2 + b^2)(a + b)(a - b)$;

$a^6 - b^6 \equiv (a^3 + b^3)(a^3 - b^3)$ (for factors of $a^3 - b^3$ see § 212);

$a^8 - b^8 \equiv (a^4 + b^4)(a^4 - b^4) \equiv (a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$;

and so on.

211. It has been shown in § 173 that the *sum* of the *even* powers of two numbers is not divisible by their sum or difference, and it will be found by trial that the sum of the *even* powers

of two numbers is not divisible by the sum or difference of *any* powers of the numbers except when the *exponents* of such even powers contain an *odd* factor.

Thus, a^2+b^2, a^4+b^4, a^8+b^8, and so on, are not divisible by the sum or difference of any powers of a and b; but $a^6+b^6=(a^2)^3+(b^2)^3$, and $a^{10}+b^{10}=(a^2)^5+(b^2)^5$, and are each divisible by a^2+b^2, and hence can be factored. Thus,

$$a^6+b^6=(a^2+b^2)(a^4-a^2b^2+b^4);$$
$$a^{10}+b^{10}=(a^2+b^2)(a^8-a^6b^2+a^4b^4-a^2b^6+b^8).$$

212. It follows, conversely, from §§ 170, 172, that

$$a^3+b^3=(a+b)(a^2-ab+b^2);$$
$$a^3-b^3=(a-b)(a^2+ab+b^2);$$
$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4);$$
$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4);$$

and generally that the *sum* of any two like *odd* powers of a and b may be factored *by dividing such sum by* $a+b$; and the *difference* of any two like odd powers of a and b, *by dividing such difference by* $a-b$.

213. In like manner many binomials whose terms have unequal exponents may be factored provided these exponents have a *common* factor.

Thus, $x^{10}-y^{15}=(x^2)^5-(y^3)^5$, hence has the factor x^2-y^3; and $m^{20}-n^{12}=(m^5)^4-(n^3)^4$ or $(m^{10})^2-(n^6)^2$, and hence has as a factor either m^5-n^3, or m^5+n^3, or $m^{10}-n^6$, or $m^{10}+n^6$.

1. Resolve into factors (1) x^3+y^3; (2) x^3-y^3.

(1) $x^3+y^3=(x+y)(x^2-xy+y^2)$.
(2) $x^3-y^3=(x-y)(x^2+xy+y^2)$.

2. Resolve into factors x^6-y^6.

$$x^6-y^6=(x^3+y^3)(x^3-y^3);$$
$$x^3+y^3=(x+y)(x^2-xy+y^2);$$
$$x^3-y^3=(x-y)(x^2+xy+y^2).$$
$$\therefore\ x^6-y^6=(x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2).$$

3. Resolve into factors $8a^3b^3 - 125c^6$.

$$\sqrt[3]{8a^3b^3} = 2ab\,;\quad \sqrt[3]{125c^6} = 5c^2.$$

$\therefore$ (dividing $8a^3b^3 - 125c^6$ by $2ab - 5c^2$)

$$8a^3b^3 - 125c^6 = (2ab - 5c^2)(4a^2b^2 + 10abc^2 + 25c^4).$$

4. Resolve into factors $(a-b)^3 - 8c^3$.

$$\sqrt[3]{(a-b)^3} = a - b\,;\quad \sqrt[3]{8c^3} = 2c.$$

$\therefore$ (dividing $\overline{a-b}^3 - 8c^3$ by $\overline{a-b} - 2c$)

$$(a-b)^3 - 8c^3 = (a-b-2c)[(a-b)^2 + (a-b)2c + 4c^2].$$

Resolve into factors

5. $x^3 + 8$.	**14.** $m^6 - n^6$.
6. $x^3 - 1$.	**15.** $m^6 + n^6$.
7. $1 + x^3$.	**16.** $x^5 - 1$.
8. $x^3 - 27$.	**17.** $1 + (a+b)^3$.
9. $125 + x^6$.	**18.** $(a-b)^3 + 1$.
10. $16 - x^4$.	**19.** $(m+n)^4 - (m-n)^4$.
11. $x^4 - 256$.	**20.** $(m+n)^3 - (m-n)^3$.
12. $a^6 - 27$.	**21.** $(2a+x)^4 - (a+2x)^4$.
13. $x^6 - y^3$.	**22.** $8a^3 - (a+2x)^3$.

214. The sum of any even powers of two numbers, as a^2+b^2, a^4+b^4, a^8+b^8, $(a^{2n}+b^{2n})$, may be factored by adding to the binomial such a number as will make it a perfect square, and then subtracting the number added from the result, as shown in Case V., §§ 194, 195.

Thus, $\quad a^4 + b^4 = a^4 + b^4 + 2a^2b^2 - 2a^2b^2$

$$= (a^2+b^2)^2 - 2a^2b^2 = (a^2 + b^2 + \sqrt{2a^2b^2})(a^2 + b^2 - \sqrt{2a^2b^2}).$$

Resolve into factors

23. $x^2 + y^2$.	**27.** $a^8 + b^8$.
24. $a^2 + b^4$.	**28.** $9x^4 + 4y^2$.
25. $m^4 + n^8$.	**29.** $4x^2 + y^4$.
26. $81 + x^4$.	**30.** $x^4 + 256$.

FACTORING BY SYNTHETIC DIVISION.

215. It will be found by multiplying together several polynomial factors, arranged according to a common letter or letters, that the *first* term of the product is *the product of the first terms of the factors*, and that the *last* term of the product is *the product of the last terms of the factors.*

Thus, $(x-2)\,(x+3)\,(x-5) = x^3 - 4x^2 - 11x + 30$; and $(x+a)\,(x-b)\,(x+c) = x^3 + (a-b+c)\,x^2 - (ab - ac + bc)\,x - abc$.

216. It follows that a polynomial which contains one or more binomial factors may be factored by resolving its first and last terms into factors, and then determining *by synthetic division* (§ 132) which of the binomials that can be formed by uniting these factors (two and two) with the sign + or − are *factors of the given polynomial.*

This process is made plain below by the factoring of a few polynomials, and a little practice will enable pupils to use it readily.

1. Resolve into factors $x^3 - 4x^2 - 11x + 30$.

The three factors of x^3 are x, x, and x, and the three positive factors of 30 are 2, 3, and 5, or 1, 5, and 6, or 1, 2, and 15. Since two of the terms of the polynomial are negative, and the last term positive, try successively by synthetic division the divisors $x-2$, $x+3$, and $x-5$.

1	$1 - 4 - 11 + 30$	
$+2$	$+2$	
	-4	
	-30	
1	$1 - 2 - 15$	($x^2 - 2x - 15$, first quotient)
-3	-3	
	$+15$	
	$1 - 5$	($x - 5$, second quotient)

Hence the factors sought are $x-2$, $x+3$, and $x-5$.

If we had tried as divisors $x+1$ or $x-1$ or $x+2$ or $x-3$ or $x+5$, there would have been in each case a remainder, thus showing that the trial divisor is not a factor.

In the foregoing process it was assumed that the polynomial $x^3-4x^2-11x+30$ is composed of three binomial factors; but it may be found on trial that a polynomial of the third degree contains only one rational binomial factor, the other factor being a trinomial, as in Example 5.

2. Resolve into factors $x^3-43x+42$.

Try $x-1$, $x-6$, and $x+7$.

1 / +1	1 − 0 − 43 + 42 + 1 + 1 − 42
1 / +6	1 + 1 − 42 (x^2+x-42, 1st quo.) + 6 + 42
	1 + 7 ($x+7$, 2d quo.)

The process at the left may be condensed by writing the several partial products in a horizontal line, thus:

1 / +1	1 − 0 − 43 + 42 + 1 + 1 − 42
1 / +6	1 + 1 − 42 (x^2+x-42, 1st quo.) + 6 + 42
	1 + 7 ($x+7$, 2d quo.)

The factors sought are $x-1$, $x-6$, and $x+7$.

3. Resolve into factors $x^4-10x^3+32x^2-38x+15$.

First try in succession $x-1$, $x-3$, $x-5$.

1 / +1	1 − 10 + 32 − 38 + 15 + 1 − 9 + 23 − 15
1 / +3	1 − 9 + 23 − 15 ($x^3-9x^2+23x-15$, 1st quo.) + 3 − 18 + 15
1 / +5	1 − 6 + 5 (x^2-6x+5, 2d quo.) + 5 − 5
	1 − 1 ($x-1$, 3d quo.)

The factors sought are $x-1$, $x-3$, $x-5$, and $x-1$.

4. Resolve into factors $2x^3+31x^2+62x-39$.

Try in succession $2x-1$, $x+3$, $x+13$.

2 / +1	2 + 31 + 62 − 39 + 1 + 16 + 39
1 / −3	1 + 16 + 39 ($x^2+16x+39$) − 3 − 39
	1 + 13 ($x+13$)

The factors are $2x-1$, $x+3$, and $x+13$.

5. Resolve into factors $3x^3 - 46x^2 + 9x + 2$.

First try $3x - 1$.

$$\begin{array}{r|l} 3 & 3 - 46 + 9 + 2 \\ +1 & \quad + 1 - 15 - 2 \\ \hline & 1 - 15 - 2 \end{array} \qquad (x^2 - 15x - 2)$$

If we try $x - 1$ or $x + 1$ or $x - 2$ or $x + 2$, we shall have in each case a remainder, and so conclude that $x^2 - 15x - 2$ cannot be resolved into rational factors. This is also made obvious by the fact that the coefficient of the second term (-15) is not the algebraic sum of any two rational factors of -2 (§ 371).

Hence the factors sought are $3x - 1$ and $x^2 - 15x - 2$.

6. Resolve into factors $x^5 - 3x^4 - 18x^3 + 87x^2 - 109x + 30$.

Try first successively $x - 2$, $x - 3$, and $x + 5$.

$$\begin{array}{r|l} 1 & 1 - 3 - 18 + 87 - 109 + 30 \\ +2 & \quad + 2 - 2 - 40 + 94 - 30 \\ \hline 1 & 1 - 1 - 20 + 47 - 15 \\ +3 & \quad + 3 + 6 - 42 + 15 \\ \hline 1 & 1 + 2 - 14 + 5 \\ -5 & \quad - 5 + 15 - 5 \\ \hline & 1 - 3 + 1 \end{array} \qquad (x^2 - 3x + 1)$$

It will be found by trial that neither $x - 1$ nor $x + 1$ is a factor of $x^2 - 3x + 1$; and it is also evident on inspection that the trinomial has no rational factors.

Resolve into factors

7. $x^3 + 4x^2 + x - 6$.
8. $x^3 - 7x - 6$.
9. $x^3 - 19x - 30$.
10. $x^3 - 43x + 42$.
11. $x^3 - 10x^2 + 31x - 30$.
12. $x^3 + 9x^2 + 11x - 21$.
13. $x^3 - x^2 - 8x + 12$.
14. $3x^3 - 10x^2 - x + 12$.
15. $2x^3 - 9x^2 + 13x - 6$.
16. $6x^3 + 7x + 13$.
17. $2x^3 - 5x + 39$.
18. $x^4 - 4x^3 - 7x^2 + 34x - 24$.
19. $x^4 + x^3 - 4x - 16$.
20. $x^4 - 3x^3 - 3x^2 + 12x - 4$.

67. $x^6 - 64$.
68. $1 - m^6$.
69. $27 - x^3$.
70. $1 + (x + y)^3$.
71. $(x - y)^3 + 1$.
72. $x^4 - 12x^2 + 11$.
73. $x^4 - 8x^2 + 7$.
74. $a^4 - 10a^2 + 9$.
75. $x^2 + 2x - 120$.
76. $x^2 + 21x - 72$.
77. $x^2 + x - 240$.
78. $x^2 - 10x - 144$.
79. $x^2 + 30x + 221$.
80. $21x^2 - 4x - 1$.
81. $36x^2 - 31x - 56$.
82. $x^2 - 41x + 420$.
83. $x^2 - 27x + 180$.
84. $x^3 - x^2 - 20x$.
85. $4a^2 - 14a + 6$.
86. $4a^4 - 10a^2 + 4$.
87. $3x^2 - 2x - 65$.
88. $3x^2 - 6x - 15$.
89. $x^2 - 6x - 7$.
90. $x^2 - 10x + 25$.
91. $x^2 - 5x - 14$.
92. $x^4 - 7x^2 - 60$.
93. $x^4 - 6x^2 - 40$.
94. $81x^4 - 72x^2y^2 + 16y^4$.
95. $x^4 + m^2x^2 - n^2x^2 - m^2n^2$.
96. $x^2 + 2ax + 2x + (a + 1)^2$.
97. $a^4 - a^3y + ay^3 - y^4$.
98. $x^4 - (3m - 2)x^2 - 6m$.
99. $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$.
100. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$.
101. $m^2 + n^2 - a^2 - b^2 - 2mn + 2ab$.
102. $4 - 4a + 2mn - m^2 + a^2 - n^2$.
103. $9x^2 + y^2 - 4z^2 - 6xy - 8sz - 4s^2$.
104. $x^2 + y^2 + 1 - 2xy + 2x - 2y$.
105. $ax^3 - by^2 + 2ax^2y - bx^2 + axy^2 - 2bxy$.
106. $a^2x^4 - 2abx^4 + b^2x^4 - a^2y^4 + 2aby^4 - b^2y^4$.
107. $mx^2 + 2nxy - ny^2 - 2mxy - nx^2 + my^2$.
108. $x^3 - 9x^2 + 23x - 15$.
109. $x^4 - 2x^3 - 16x^2 + 2x + 15$.
110. $x^4 + 7x^3 + 9x^2 - 7x - 10$.

CHAPTER VIII.

COMMON FACTORS AND MULTIPLES.

COMMON FACTORS.

218. Any number is divisible by any one of its factors or by the product of any two or more of its prime factors. Thus, since $30 = 2 \times 3 \times 5$, 30 is divisible by 2 or 3 or 5, or by 2×3 or 2×5 or 3×5. In like manner $3ab$ is divisible by 3 or a or b or by $3a$ or $3b$ or ab.

219. Two or more numbers are divisible by any common factor or by the product of any two or more common prime factors. Thus, since $12 = 2 \times 2 \times 3$, and $30 = 2 \times 3 \times 5$, 12 and 30 are divisible by 2 or 3 or by 2×3. In like manner $3a^2b$ and $6abc^2$ are divisible by 3 or a or b, or by $3a$ or $3b$ or ab or $3ab$.

220. It follows that a common factor of two or more numbers is their **common divisor.**

Two or more numbers which have no common integral factor except 1 are *prime with respect to each other.*

Highest Common Factor.

221. The **highest common factor** of two or more numbers is the factor of highest degree that will divide each number without a remainder. Thus, $\pm 3a^2b$ is the highest common factor of $6a^3bx$ and $9a^2b^2y$, and $\pm 4a^3b^2$ is the highest common factor of $12a^4b^3$ and $-16a^3b^2$.

In arithmetic the *greatest* number that will exactly divide two or more numbers is called their *greatest* common divisor; but since, in algebra, a

negative number is less than any positive number or than a negative number that is numerically smaller (§ 67), the highest common factor of two or more numbers may be *algebraically less* than any other factor, positive or negative. Thus, $-3a^2b < \pm a$ or $\pm ab$. This explains and justifies the use of the expression "greatest common factor" in arithmetic, and "highest common factor" in algebra.

222. The highest common factor of two or more algebraic numbers is their **highest common divisor.**

For convenience the abbreviation H. C. F. is used in this chapter for "highest common factor," and H. C. D. for "highest common divisor."

H. C. F. found by Factoring.

223. The H. C. F. or H. C. D. of two or more algebraic numbers may be found *by resolving each into its prime factors, and then taking the product of all those factors which are common.*

The numbers in the following exercises may be readily resolved into factors *by inspection.*

1. Find the H. C. F. of $63a^2x^3y$ and $42abx^2z^2$.

$$63a^2x^3y = 3 \times 3 \times 7a^2x^3y;$$
$$42abx^2z^2 = 2 \times 3 \times 7abx^2z^2.$$
$$\therefore 3 \times 7ax^2, \text{ or } 21ax^2, \text{ is the H. C. F. required.}$$

The H. C. F. of two or more numbers may be either positive or negative, and hence $\pm 21ax^2$ is the H. C. F. of $63a^2x^3y$ and $42abx^2z^2$. In this chapter only the positive H. C. F. is given or required.

2. Find the H. C. F. of $x^3 - xy^2$ and $x^3 - 2x^2y + xy^2$.

$$x^3 - xy^2 = x(x^2 - y^2) = x(x + y)(x - y);$$
$$x^3 - 2x^2y + xy^2 = x(x - y)(x - y).$$
$$\therefore x(x - y), \text{ or } x^2 - xy, \text{ is the H. C. F. required.}$$

3. Find the H. C. F. of $x^2 - 12x + 35$, $x^2 - 10x + 25$, and $x^2 + 6x - 55$.

$$x^2 - 12x + 35 = (x - 5)(x - 7);$$
$$x^2 - 10x + 25 = (x - 5)(x - 5);$$
$$x^2 + 6x - 55 = (x - 5)(x + 11).$$
$$\therefore x - 5 \text{ is the H. C. F. required.}$$

4. Find the H. C. F. of $a^4 - 2a^2b^2 + b^4$ and $a^4 - b^4$.

$a^4 - 2a^2b^2 + b^4 = (a^2 - b^2)^2 = (a^2 - b^2)(a^2 - b^2)$;
$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$.
$\therefore$ $a^2 - b^2$ is the H. C. F. required.

5. Find the H. C. F. of $3x^2 + 4x - 15$ and $5x^2 - 56x + 11$.

$3x^2 + 4x - 15 = (3x - 5)(x + 3)$;
$5x^2 - 56x + 11 = (5x - 1)(x - 11)$.
There is no common factor, and hence no H. C. F.

Find the H. C. F. of

6. 210 and 330; 450, 625, and 825.
7. $21a^3b^2$ and $63ab^3c$; $75a^3b^3c$ and $60ab^2c^3$.
8. $24x^2y^2z^2$, $16x^4y^5$, and $72ax^3y^3z^3$.
9. $30a^5x^3y^4$, $24a^3x^2y^3$, $72a^3x^3y^6$, and $36a^4x^2y^4$.
10. $x^2 - y^2$, $x^3 - y^3$, and $x^2 - 2xy + y^2$.
11. $x^2 - 1$, $(x - 1)^2$, and $x^3 - 1$.
12. $x^6 + a^3x^3$ and $x^4 - a^2x^2$.
13. $15(x - 1)^3$ and $45(x^4 - 1)$.
14. $4(a^6 - b^6)$ and $20(a^4 - b^4)$.
15. $a^3 + b^3$ and $a^3 - a^2b + ab^2$.
16. $x^2 - 2x + 1$ and $x^4 - 1$.
17. $x^2 + x + 1$ and $x^4 + x^2 + 1$.
18. $x^4 - 8x^2 + 16$ and $3x^3 - 12x^2 + 12x$.
19. $10a^2y^2 + 40a^2yz + 40a^2z^2$ and $y^4 - 16z^4$.
20. $a^2x^4 - a^2y^2$ and $x^5 + x^3y$.
21. $x^2 + 8x + 16$ and $x^4 - 256$.
22. $x^2 - 10x + 9$, $x^2 - 17x + 72$, and $x^2 - 11x + 18$.
23. $x^2 + 10x + 21$, $x^2 + 4x - 21$, and $x^2 - x - 56$.
24. $x^2 - 9x - 36$, $2x^2 - 30x + 72$, and $3x^2 - 42x + 72$.
25. $3x^2 - 4x - 7$, $5x^2 + 3x - 2$, and $15x^2 + 18x + 3$.
26. $x^4 + x^2 - 132$ and $x^4 + 17x^2 + 60$.

27. $x^4 - 11x^3 + 10x^2$ and $x^3 + 7x - 8$.

28. $x^3 - 18x + 65$ and $x^3 + 2x - 35$.

29. $6x^3 + x - 12$ and $4x^3 + 12x + 9$.

30. $x^3 - 1$, $x^4 - 1$, and $x^5 - 2x^3 + x$.

H. C. F. found by Continued Division.

224. When two or more polynomials are not readily resolved into factors by inspection, their H. C. F. may be found by the method of *continued division*, a method similar to the corresponding one in arithmetic.

225. This method depends on simple principles: to wit,

I. *The factor of a number is the factor of any multiple of the number.* Thus, 6, a factor of 12, is a factor of 12×2, $12 \times 3 \cdots 12 \times n$.

II. *A common factor of two numbers is a factor of their sum or of their difference, also of the sum or difference of any of their multiples.* Thus, 6, a common factor of 24 and of 18, is a factor of $24 + 18$ or $24 - 18$, also of $24 \times 2 + 18 \times 2$ or $24 \times 2 - 18 \times 2$. Hence

III. *A common factor of either of two numbers and their difference is a common factor of the two numbers.* Thus, 6, a common factor of 18 and of $30 - 18$ (or 12), is a common factor of 18 and 30.

1. Find the H. C. F. of 288 and 816.

$$
\begin{array}{l}
288)816(2 \\
\quad \underline{576} \\
\quad 240)288(1 \\
\quad\quad \underline{240} \\
\quad\quad 48)240(5 \\
\quad\quad\quad \underline{240}
\end{array}
$$

48 is the H. C. F. of 288 and 816.

Since 48 is a common factor of 48 and 240, it is a factor of 288, their sum (II.).

Since 48 is a factor of 288, it is a factor of 288×2, or 576 (I.).

Since 48 is a common factor of 240 and 576, it is a factor of 816, their sum (II.).

Hence 48 is a *common factor* of 288 and 816.

Since any common factor of 288 and 816 is a factor of 240 (II.), and hence a common factor of 288 and 240 and also of 240 and 48, it follows that 48, the *greatest* common factor of 240 and 48, is the H. C. F. of 288 and 816.

226. The following is a general explanation of this method:

Let a and b represent the two numbers, and q, q', q'', and r, r', r'', and so on, denote respectively the successive quotients and remainders, and suppose that $r'' = 0$. Thus,

$$
\begin{array}{l}
b)a(q \\
\underline{bq} \\
\quad r)b(q' \\
\quad \underline{rq'} \\
\quad\quad r')r(q'' \\
\quad\quad \underline{r'q''} \\
\quad\quad\quad r'' = 0
\end{array}
$$

Since r' is a common factor of r' and r, it is a factor of rq' (§ 225, I.), and hence of b, the sum of r' and rq' (II.).

Since r' is a common factor of r and b, it is a factor of bq (I.), and hence of a, the sum of bq and r (II.).

Hence r' is a common factor of a and b.

Since any common factor of a and b is a factor of r, and since any common factor of b and r is a factor of r', it follows that the *greatest* common factor of r and r' (which is r') is the H. C. F. of a and b.

2. Find the H. C. F. of $6x^3-7x^2-22x+32$ and $3x^2+x-10$.

$$\begin{array}{l|l}
6x^3-7x^2-22x+32 & 3x^2+x-10 \\
\underline{6x^3+2x^2-20x} & \underline{2x-3} \\
-9x^2-2x+32 & \\
\underline{-9x^2-3x-30} & \\
3x^2+x-10 & x+2 \\
\underline{3x^2+6x} & \underline{3x-5} \\
-5x-10 & \\
\underline{-5x-10} & \therefore\ x+2 \text{ is the H. C. F. required.}
\end{array}$$

3. Find the H. C. F. of $a^4-a^3b+ab^3-b^4$ and $a^3-a^2b-ab^2+b^3$.

$$\begin{array}{l|l}
a^4-a^3b+ab^3-b^4 & a^3-a^2b-ab^2+b^3 \\
\underline{a^4-a^3b-a^2b^2+ab^3} & \underline{a} \\
\multicolumn{2}{c}{a^2b^2-b^4=(a^2-b^2)b^2} \\
a^3-a^2b-ab^2+b^3 & a^2-b^2 \\
\underline{a^3-ab^2} & \underline{a-b} \\
-a^2b+b^3 & \\
\underline{-a^2b+b^3} & \therefore\ a^2-b^2 \text{ is the H. C. F. required.}
\end{array}$$

227. It is seen that $a^2b^2-b^4$, the first remainder, will not divide the first divisor; but $a^2b^2-b^4=(a^2-b^2)b^2$, and, removing the b^2, we obtain a^2-b^2, which is a divisor. Since b^2 is not a factor of $a^3-a^2b-ab^2+b^3$, it is evident that it is not a factor of the H. C. F., and hence may be set aside. At any stage of the process, a factor of dividend or divisor, not common to both, may be rejected.

4. Find the H. C. F. of $3x^3+x^2-4x-10$ and $9x^2-9x-10$.

Multiply by 3,
$$\begin{array}{l|l}
3x^3+x^2-4x-10 & \\
\underline{9x^3+3x^2-12x-30} & 9x^2-9x-10 \\
9x^3-9x^2-10x & \underline{x+4}
\end{array}$$

Multiply by 3,
$$\begin{array}{r}
12x^2-2x-30 \\
\underline{36x^2-6x-90}
\end{array}$$

$(9x^2-9x-10)\times 4 =$
$$\begin{array}{r}
\underline{36x^2-36x-40} \\
30x-50
\end{array}$$

Divide by 10,
$$\begin{array}{l|l}
9x^2-9x-10 & 3x-5 \\
\underline{9x^2-15x} & \underline{3x+2} \\
6x-10 & \\
\underline{6x-10} & \therefore\ 3x-5 \text{ is the H. C. F. required.}
\end{array}$$

5. Find the H. C. F. of $10abx^3 + 10abx^2 - 90abx - 90ab$ and $6ax^4 - 42ax^2 - 36ax$.

$10abx^3 + 10abx^2 - 90abx - 90ab = 10ab(x^3 + x^2 - 9x - 9)$;

$6ax^4 - 42ax^2 - 36ax = 6ax(x^3 - 7x - 6)$;

$10ab = 2a \times 5b$; $6ax = 2a \times 3x$. $\therefore$ $2a$ is common.

$$\begin{array}{r|l} x^3 + x^2 - 9x - 9 & x^3 - 7x - 6 \\ x^3 \quad - 7x - 6 & 1 \\ \hline x^3 - 7x - 6 & x^2 - 2x - 3, \text{ common.} \\ x^3 - 2x^2 - 3x & x + 2 \\ \hline 2x^2 - 4x - 6 & \\ 2x^2 - 4x - 6 & \\ \hline \end{array}$$

$\therefore$ $2a(x^2 - 2x - 3)$ is the H. C. F. required.

228. At any stage of the process, a divisor or dividend may be multiplied or divided by any factor that is not common to both, as in Example 4; and any common factor may be set apart as a factor of the H. C. F., as $2a$ in Example 5.

229. To find the H. C. F. of two polynomials by continued division,

Remove the monomial factor, if any, from each polynomial, and set aside the common factor in the same, if any, as a factor of the H. C. F.

Arrange the resulting polynomials in descending powers of some common letter, and divide the polynomial of the higher degree by the other, and then the divisor by the remainder (if any), and the second divisor by the second remainder, and so on until there is no remainder. The last divisor is the H. C. F. of the first dividend and divisor, and the product of this factor and the common monomial factor, if any, is the H. C. F. required.

If the two polynomials are of the same degree of the common letter, either may be used as the divisor.

230. To find the H. C. F. of more than two polynomials,

First find the H. C. F. of two of them, and then the H. C. F. of this result and a third polynomial, and so on.

Find the H. C. F. of

6. $x^2 - 18x + 65$ and $x^3 - 18x - 35$.

7. $6x^2 + x - 12$ and $6x^3 + 7x^2 - x + 3$.

8. $x^3 + 125$ and $2x^3 + 7x^2 + 75$.

9. $x^3 - 10x - 24$ and $x^3 - 23x + 28$.

10. $x^3 - 16x + 21$ and $x^3 + 7x - 48$.

11. $x^3 + 15x - 306$ and $x^3 - 26x - 60$.

12. $4x^3 - 21x^2 + 15x + 20$ and $x^2 - 6x + 8$.

13. $2x^3 - 120x + 378$ and $5x^3 - 42x^2 + 78x + 63$.

14. $x^4 - x^3 + 2x^2 + x + 3$ and $x^4 + 2x^3 - x - 2$.

15. $x^3 + 3x^2 + 4x + 12$ and $x^3 + 4x^2 + 4x + 3$.

16. $4x^4 - 8x^3 - 20x^2 + 24x + 20$ and $3x^3 + 6x^2 - 24x - 45$.

17. $x^4 - 5x^3 + 8x^2 - 7x + 3$ and $2x^3 - 9x^2 + 10x - 3$.

18. $18x^3 - 51x^2 + 13x + 5$ and $6x^2 - 13x - 5$.

19. $2x^4 - 3x^3 + 2x^2 - 2x - 3$ and $6x^3 - x^2 + 8x + 3$.

20. $x^3 - x^2y - xy^2 + y^3$ and $x^3 + x^2y - xy^2 - y^3$.

The H. C. F. of the polynomials in the foregoing examples may be readily found by synthetic division (§ 216). It is recommended that the polynomials in several of the foregoing examples be thus resolved into factors, and their H. C. F. found.

COMMON MULTIPLES.

Lowest Common Multiple.

231. A **multiple** of a number is the product of the number multiplied by an integer; and hence any multiple of a number is exactly divisible by the number. 12 is a multiple of 2, 3, 4, and 6; and $3ab$ is a multiple of 3, a, and b, also of $3a$, $3b$, and ab.

The product of two or more integral factors is a multiple of each factor.

232. A **common multiple** of two or more numbers is a multiple of each of them. Thus, 12 is a common multiple of 2, 3, 4, and 6; and $6a^2b^2$ is a common multiple of 6, a^2, b^2, $3a^2$, $3b^2$, etc.; and $x^2 - y^2$ is a common multiple of $x + y$ and $x - y$.

233. The **lowest common multiple** of two or more numbers is the lowest multiple of each of them. Thus, 12 is the lowest common multiple of 3 and 4, and $x^2 - y^2$ is the lowest common multiple of $x + y$ and $x - y$. Hence

234. *A common multiple of two or more numbers is exactly divisible by each of them; and the lowest common multiple of two or more numbers is the lowest number that is exactly divisible by each of them.*

For convenience the abbreviation L. C. M. is used for "lowest common multiple."

The L. C. M. found by Factoring.

235. Since any multiple of a number contains all of its factors, a common multiple of two or more numbers contains all the prime factors of each; and *the L. C. M. of two or more numbers contains all the prime factors of each number, and each factor in the highest degree in which it occurs.*

Thus, the L. C. M. of $3a^2b^4$ and $a^3b^2c^2$ is $3 \times a^3 \times b^4 \times c^2$.

1. Find the L. C. M. of $6a^2bx^2$ and $10ab^2x^3y^2$.

$$6a^2bx^2 = 2 \times 3a^2bx^2;$$
$$10ab^2x^3y^2 = 2 \times 5ab^2x^3y^2.$$

$\therefore$ $2 \times 3 \times 5a^2b^2x^3y^2$, or $30a^2b^2x^3y^2$, is the L. C. M. required.

2. Find the L. C. M. of $x^2 - y^2$, $x^2 - 2xy + y^2$, and $x^2 + 2xy + y^2$.

$$x^2 - y^2 = (x + y)(x - y);$$
$$x^2 - 2xy + y^2 = (x - y)^2;$$
$$x^2 + 2xy + y^2 = (x + y)^2.$$

$\therefore$ $(x + y)^2 \times (x - y)^2$, or $x^4 - 2x^2y^2 + y^4$, is the L. C. M. required.

3. Find the L. C. M. of x^3-9, $15x^2-39x-18$, and $3ax^2-18ax+27a$.

$$x^3-9=(x+3)(x-3);$$
$$15x^2-39x-18=3(x-3)(5x+2);$$
$$3a^2x^2-18ax+27a=3a(x-3)^2.$$
$$\therefore\ 3a(x-3)^2(x+3)(5x+2) \text{ is the L. C. M. required.}$$

Find by factoring the L. C. M. of

4. $5a^3x$ and $15a^2x^4y$.

5. $14x^2yz^3$ and $63xy^3z^2$.

6. $6ab^2cd^5$, $9a^3b^3c^3$, $15ab^2c^3$, and $20a^5c^2dx^2$.

7. x^2-y^2 and x^3-y^3.

8. $7x(a+b)^2$ and $5y^2(a^3+b^3)$.

9. a^3-a^2b, a^3-b^3, and $a^2-5ab+4b^2$.

10. $x^2-7x+10$ and $x^2+3x-10$.

11. $x^2-x-110$, x^2+x-90, and $2x^2+15x-50$.

12. $x^3+8x+15$ and x^2+4x-5.

13. $x^2-5x-84$ and $x^2-7x-60$.

14. $7x^2-36x+5$ and x^2-25.

15. a^2+ab, $ab-b^2$, and a^2-b^2.

16. x^4-y^4, x^2-y^2, x^2+y^2, and $x^4-2x^2y^2+y^4$.

17. x^3-1, x^3+1, x^2-1, and x^6-1.

The L. C. M. found by the H. C. F.

236. When the factors of two polynomials cannot be found by inspection, they may each be resolved into two factors *by finding their H. C. F. by continued division, and then dividing each by this H. C. F.*

237. The H. C. F. of two polynomials contains *all their common factors,* and the two quotients obtained by dividing each polynomial by the H. C. F. contain *all the factors not common;* and hence *the product of their H. C. F. and the two quotients thus obtained will be the L. C. M. of the two polynomials.* It contains all of their factors, and each in the highest degree.

The process may be somewhat abridged *by dividing one of the polynomials by their H. C. F., and multiplying the quotient by the other polynomial.*

1. Find the L. C. M. of $x^3 - x^2y - xy^2 + y^3$ and $x^3 + x^2y - xy^2 - y^3$.

The H. C. F. of the polynomials is $x^2 - y^2$,

and $(x^3 - x^2y - xy^2 + y^3) \div x^2 - y^2 = x - y$.

$\therefore (x - y)(x^3 + x^2y - xy^2 - y^3)$ is the L. C. M. required.

2. Find the L. C. M. of $x^3 - 48x + 7$ and $x^3 + 8x^2 + 9x + 14$.

The H. C. F. of the polynomials is $x + 7$,

and $(x^3 - 48x + 7) \div (x + 7) = x^2 - 7x + 1$.

$\therefore (x^2 - 7x + 1)(x^3 + 8x^2 + 9x + 14)$ is the L. C. M. required.

Find the H. C. F. and the L. C. M. of

3. $12x^2 - 4x - 21$ and $6x^3 - 17x + 12$.
4. $x^4 - 23x^2 - 10x$ and $7x^3 - 34x^2 - 4x - 5$.
5. $3x^3 - 8x^2 - 12x - 1$ and $5x^3 - 12x^2 - 24x - 7$.
6. $3x^3 - 5x + 2$ and $21x^3 - 20x^2 - 11x + 10$.
7. $4x^3 - 3x^2 + 16x - 12$ and $12x^3 - 17x^2 + 22x - 12$.
8. $x^5 - 2x^3 - x^2$ and $x^3 + 2x^2 + 2x + 1$.
9. $x^3 - 9x^2 + 26x - 24$ and $x^3 - 6x^2 + 11x - 6$.
10. $6x^2 - 13x + 6$, $2x^2 + 5x - 12$, and $6x^2 - x - 12$.

238. The L. C. M. of the polynomials in the above examples may be readily found by synthetic division, the factors thus obtained being united as in § 235.

CHAPTER IX.

FRACTIONS.

239. A **fraction** is one or more of the equal parts of a unit.

The unit which is divided into equal parts is the **unit of the fraction,** and one of these equal parts is the **fractional unit.**

An integer is one or more integral units, and a fraction one or more fractional units.

240. A fraction is expressed by two numbers,—one called the **denominator,** denoting the number of equal parts into which the unit is divided; and the other called the **numerator,** denoting the number of equal parts taken.

241. In arithmetic, fractions are expressed by words or figures. When expressed by figures in the common form, the numerator is written above, and the denominator below, a horizontal line, as $\frac{5}{7}$. In expressing decimal fractions, the denominator need not be written, but may be indicated by the decimal point. Thus, seven tenths is written $\frac{7}{10}$ or .7.

In algebra, fractions are expressed by algebraic symbols, the numerator being written above the denominator, as in arithmetic. Thus, if a unit is supposed to be divided into b equal parts, and a of these parts are taken, the resulting fraction is expressed by $\frac{a}{b}$.

In both arithmetic and algebra the numerator and denominator may be separated by an oblique line (called the *solidus*), as $5/6$ and a/b (§ 20).

242. In algebra a fraction is treated as an *indicated division*, the numerator being the dividend, and the denominator the divisor. Hence

243. An **algebraic fraction** is the quotient indicated by writing the dividend over the divisor, with a line between them.

Thus, the fraction $\frac{a}{b}$ is the quotient of a divided by b, and is read "a divided by b."

244. The numerator and denominator are called the **terms** of a fraction. Thus, a and $a+b$ are the terms of $\frac{a}{a+b}$.

245. Since $a \div b = ac \div bc$, $\frac{a}{b} = \frac{ac}{bc}$; and since $ac \div bc = a \div b$, $\frac{ac}{bc} = \frac{a}{b}$. Hence

Both terms of a fraction may be multiplied or divided by the same number without altering the value of the fraction. Hence

246. *The same factor in both terms of a fraction may be canceled, or the same factor may be inserted in both terms, without altering the value of the fraction.*

247. Since $\frac{a}{b} = \frac{a \times (-1)}{b \times (-1)} = \frac{-a}{-b}$, and $\frac{-a}{-b} = \frac{-a \times (-1)}{-b \times (-1)} = \frac{a}{b}$, it follows that *the signs of both terms of a fraction may be changed without altering the value of the fraction.*

248. $\frac{a}{b}$ or $\frac{-a}{-b} = +\frac{a}{b}$, and $\frac{+a}{-b}$ or $\frac{-a}{+b} = -\frac{a}{b}$; and hence *a fraction is positive when the signs of both its terms are alike, and negative when the signs of both its terms are unlike.* Hence

249. *The changing of the sign of either term of a fraction changes the sign of the fraction.*

Multiply both terms of

1. $\frac{1}{4}$, $\frac{5}{12}$, $\frac{a}{5}$, $\frac{6}{a}$, $\frac{x}{y}$, and $\frac{x}{12a}$ by 3.

2. $\frac{2}{3}$, $\frac{1}{5}$, $\frac{a}{b}$, $\frac{1}{m^2}$, $\frac{n^2}{m}$, and $\frac{m}{3}$ by m.

3. $\frac{3}{a+b}$, $\frac{a+1}{5}$, $\frac{1+n}{m+1}$, $\frac{x-y}{x+y}$, and $\frac{2x}{1+y}$ by x.

Divide both terms of

4. $\frac{6}{15}$, $\frac{9}{12}$, $\frac{3a}{6}$, $\frac{6}{12a}$, $\frac{3a}{6b}$, and $\frac{3}{9m^2}$ by 3.

5. $\frac{x+y}{x^2-y^2}$, $\frac{x^2-y^2}{x^2+2xy+y^2}$, and $\frac{x^2+2xy+y^2}{x+y}$ by $x+y$.

6. $\frac{a^2-b^2}{3a^2-3ab}$, $\frac{a^3-b^3}{a^2-b^2}$, and $\frac{a^4-b^4}{a^2-2ab+b^2}$ by $a-b$.

REDUCTION OF FRACTIONS.

CASE I.

250. Fractions reduced to lowest terms.

A fraction is in its **lowest terms** when its terms contain no common factor, i.e., are prime to each other (§ 220).

1. Reduce $\frac{3a^2b^3c}{15ab^2c^3}$ to its lowest terms.

$$\frac{3a^2b^3c}{15ab^2c^3} = \frac{3ab^2c \times ab}{3ab^2c \times 5c^2} = \frac{ab}{5c^2}.$$

The common factors of the two terms are 3, a, b^2, and c, and are canceled.

2. Reduce $\frac{a^4-b^4}{a^4+2a^2b^2+b^4}$ to its lowest terms.

$$\frac{a^4-b^4}{a^4+2a^2b^2+b^4} = \frac{(a^2+b^2)(a^2-b^2)}{(a^2+b^2)(a^2+b^2)} = \frac{a^2-b^2}{a^2+b^2}.$$

The common factor is a^2+b^2, which is canceled.

3. Reduce $\frac{x^2-10x+25}{x^2-12x+35}$ to its lowest terms.

$$\frac{x^2-10x+25}{x^2-12x+35}=\frac{(x-5)(x-5)}{(x-5)(x-7)}=\frac{x-5}{x-7}.$$

4. Reduce $\frac{a^3-a^2b-ab^2+b^3}{a^4-a^3b+ab^3-b^4}$ to its lowest terms.

$$\frac{(a^3-a^2b-ab^2+b^3)\div(a^2-b^2)}{(a^4-a^3b+ab^3-b^4)\div(a^2-b^2)}=\frac{a-b}{a^2-ab+b^2}.$$

The H. C. F. of both terms, found by inspection or by continued division, is a^2-b^2; and dividing both terms by a^2-b^2 reduces the fraction to its lowest terms.

Reduce to lowest terms

5. $\frac{15\,a^2b^3c^4}{25\,ab^4c}$.

6. $\frac{108\,a^3b^2x}{144\,ab^3x^2}$.

7. $\frac{-91\,m^2pq^3x^4}{119\,mp^2q^2x^3}$.

8. $\frac{a^3-a^2b}{3\,ab}$

9. $\frac{12\,a^2xy}{9\,ax^2-24\,ay}$.

10. $\frac{14\,(a-b)^2}{63\,(a^2-b^2)}$

11. $\frac{x^2+13x+22}{x^2+2x-99}$.

12 $\frac{x^2+5x-84}{x^3-14x^2+49x}$.

13. $\frac{x^2+x-72}{4\,ax^2-48\,ax+128\,a}$.

14. $\frac{x^2-15x+54}{x^3-18x^2+101x-180}$.

15. $\frac{x^3+y^3}{x^4+x^2y^2+y^4}$.

16. $\frac{x^3-1728}{x^2-x-132}$.

17. $\frac{x^2+(a+b)x+ab}{x^2+(b+c)x+bc}$.

18. $\frac{x^2-(a-b)x-ab}{x^2+(b-c)x-bc}$.

19. $\frac{x^4-y^4}{x^6-y^6}$.

20. $\frac{a^3-ax^2}{a^4-x^4}$.

21. $\frac{x^3-ax^2}{x^2-2\,ax+a^2}$.

22. $\frac{a^3+x^3}{a^3+2\,a^2x+ax^2}$.

23. $\frac{3\,x^3+36\,x^2+105\,x}{12\,x^3+132\,x^2+360\,x}$.

24. $\frac{2\,x^2+19\,x+35}{3\,x^2+15\,x-42}$.

25. $\dfrac{x^3-47x+14}{x^3-54x-35}$.

26. $\dfrac{2x^3-13x^2+23x-12}{7x^3-33x^2+18x+8}$.

27. $\dfrac{x^3-18x+35}{x^4-55x+126}$.

28. $\dfrac{a^3-a^2x+3ax^2-3x^3}{4a^2x-ax^2-3x^3}$.

29. $\dfrac{x^4-135x-486}{x^4-24x^2-63x-54}$.

30. $\dfrac{x^3+125}{2x^3+7x^2+75}$.

31. $\dfrac{x^3-16x+21}{x^3+7x-48}$.

32. $\dfrac{x^3+3x^2+4x+12}{x^3+4x^2+4x+3}$.

33. $\dfrac{x^4-x^3+2x^2+x+3}{x^4+2x^3-x-2}$.

34. $\dfrac{x^3-x^2y-xy^2+y^3}{x^3+x^2y-xy^2-y^3}$.

251. To reduce a fraction to its lowest terms,

Resolve both terms of the fraction into their prime factors, and cancel all the common factors; or *divide both terms of the fraction by their H. C. F.*

Case II.

252. Fractions reduced to integral or mixed numbers.

An **algebraic mixed number** is an integer and a fraction connected by the sign + or −. Thus, $a^2+\dfrac{a}{4}$ and $x^2-\dfrac{x}{x-1}$ are mixed numbers.

In arithmetic a mixed number is an integer and an *added* fraction. Thus, $6\frac{1}{2}=6+\frac{1}{2}$, and $16\frac{2}{3}=16+\frac{2}{3}$. In algebra the integral and fractional parts of a mixed number may be connected by + or −.

253. Since a fraction is an indicated division (§ 243), it is reduced to an integer or a mixed number by performing the operation indicated.

In arithmetic every improper fraction can be reduced to an integer or to a mixed number, but in algebra this can be done only when at least one term of the numerator is divisible by a term of the denominator.

Thus, $\dfrac{4a^2}{b}$ cannot be reduced; but

$$\frac{4a^2}{2a+b}=2a-b+\frac{b^2}{2a+b}, \text{ and } \frac{a^2-b^2}{a-b}=a+b.$$

1. Reduce $\frac{x^3 - ax^2 + bx}{x^2}$ to a mixed number.

$$\frac{x^3 - ax^2 + bx}{x^2} = x - a + \frac{b}{x}.$$

2. Reduce $\frac{x^2 - 2x + 5}{x - 1}$ to a mixed number.

$$\frac{x^2 - 2x + 5}{x - 1} = x - 1 + \frac{4}{x - 1}.$$

Reduce to integral or mixed numbers

3. $\frac{ax^2 - bx}{x}$.

4. $\frac{ab + x}{a}$.

5. $\frac{abc - cx^2}{ac}$.

6. $\frac{x^4 - y^4}{x^2 + y^2}$.

7. $\frac{x^2 + xy + y^2}{x + y}$.

8. $\frac{x^3 - y^3}{x - y}$.

9. $\frac{a^2 + b^2}{a - b}$.

10. $\frac{x^2 - 3x - 5}{x - 1}$.

11. $\frac{a^2 - x^2 + 3c}{a + x}$.

12. $\frac{x^2 + 3xy - 5}{x + y}$.

13. $\frac{3x^3 - 15ax^2 + a^2}{3x}$.

14. $\frac{12x^3 - 3x^2}{4x^3 - x^2 - 4x + 1}$.

15. $\frac{a^3 - 3a^2b + 3ab^2 - b^3}{a^2 - 2ab + b^2}$.

16. $\frac{x^3 - y^3 - z^3}{x - y}$.

254. To reduce an algebraic fraction to an integral or mixed number,

Divide the numerator by the denominator.

Case III.

255. Integral or mixed numbers reduced to fractions.

1. Reduce $16\frac{2}{3}$ to a fraction.

$$16\tfrac{2}{3} = \frac{16 \times 3}{3} + \frac{2}{3} = \frac{48 + 2}{3} = \frac{50}{3}.$$

2. Reduce $a - \frac{a^2}{a+b}$ to a fraction

$$a - \frac{a^2}{a+b} = \frac{a(a+b) - a^2}{a+b} = \frac{ab}{a+b}.$$

3. Reduce $x+y$ to a fraction with $x-y$ for its denominator.

$$x + y = \frac{(x+y)(x-y)}{x-y} = \frac{x^2 - y^2}{x-y}.$$

4. Reduce $a - b - \frac{a^2+b^2}{a+b}$ to a fraction.

$$a - b - \frac{a^2+b^2}{a+b} = \frac{(a-b)(a+b) - (a^2+b^2)}{a+b} = \frac{a^2 - b^2 - a^2 - b^2}{a+b} = \frac{-2b^2}{a+b}.$$

Reduce to a fraction

5. $3x + \frac{x}{y}$.

6. $5ay + \frac{4ay^2 - a^3}{y}$.

7. $1 - \frac{x-y}{x+y}$.

8. $5a - \frac{3ax - a}{x-1}$.

9. $a + x - \frac{a^2+x^2}{a+x}$.

10. $a^2 - b + \frac{a^3 - b^2}{a+b}$.

11. $x^2 - xy + y^2 - \frac{y^3}{x+y}$.

12. $(x-z)^2 + \frac{z^3}{x-z}$.

13. $3x - 2y - \frac{3x^2 - 2y^2}{x+y}$.

14. $\frac{a^2b - 2x}{a+x} - ab + 2$.

256. To reduce an integer to a fraction,

Multiply the integer by the denominator of the required fraction, and under the product write the denominator.

257. To reduce a mixed number to a fraction,

Multiply the integral part by the denominator of the fraction, and to the product add, or from the product subtract, according as the fraction is + or −, the numerator, and under the result write the denominator.

When the fraction is preceded by the sign −, as in Example 4 above, either its numerator, if a polynomial, must be inclosed in a parenthesis preceded by the sign −, or the sign of each of its terms must be changed.

Case IV.

258. Fractions reduced to their lowest common denominator.

Fractions with unlike denominators may be reduced to equivalent fractions with a common denominator, as shown below.

1. Reduce $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{7}{12}$ to equivalent fractions with their lowest common denominator.

The L. C. M. of 4, 8, and 12 is 24 (§ 234). Change the fractions to 24ths.

$$\frac{3}{4} = \frac{18}{24}; \quad \frac{5}{8} = \frac{15}{24}; \quad \frac{7}{12} = \frac{14}{24}.$$

$\therefore$ $\frac{18}{24}$, $\frac{15}{24}$, and $\frac{14}{24}$ are the equivalent fractions required.

2. Reduce $\frac{3x-6}{2a}$, $\frac{6+5x}{3a^2}$, and $\frac{3x}{4a^3}$ to equivalent fractions with their lowest common denominator.

The L. C. M. of $2a$, $3a^2$, and $4a^3$ is $12a^3$.

$$\frac{3x-6}{2a} = \frac{6a^2(3x-6)}{12a^3} = \frac{18a^2x - 36a^2}{12a^3};$$

$$\frac{6+5x}{3a^2} = \frac{4a(6+5x)}{12a^3} = \frac{24a+20ax}{12a^3};$$

$$\frac{3x}{4a^3} = \frac{3 \times 3x}{12a^3} = \frac{9x}{12a^3}.$$

$\therefore$ $\frac{18a^2x - 36a^2}{12a^3}$, $\frac{24a+20ax}{12a^3}$, and $\frac{9x}{12a^3}$ are the equivalent fractions required.

3. Reduce $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{a}$ to equivalent fractions with the lowest common denominator.

The L. C. M. of b, c, and a is abc.

$$\frac{a}{b} = \frac{a^2c}{abc}; \quad \frac{b}{c} = \frac{ab^2}{abc}; \quad \frac{c}{a} = \frac{bc^2}{abc}.$$

$\therefore$ $\frac{a^2c}{abc}$, $\frac{ab^2}{abc}$, and $\frac{bc^2}{abc}$ are the equivalent fractions required.

259. It is seen from the foregoing solution, that, when the denominators of the several fractions are prime to each other, they are reduced to like fractions *by multiplying both terms of each fraction by all the denominators except its own.*

4. Reduce $\frac{a+b}{a-b}$, $\frac{a-b}{a+b}$, and $\frac{3a}{a^2-b^2}$ to equivalent fractions with the lowest common denominator.

The L. C. M. of $a-b$, $a+b$, and a^2-b^2 is a^2-b^2.

$$\frac{a+b}{a-b}=\frac{(a+b)(a+b)}{a^2-b^2}=\frac{(a+b)^2}{a^2-b^2};$$

$$\frac{a-b}{a+b}=\frac{(a-b)(a-b)}{a^2-b^2}=\frac{(a-b)^2}{a^2-b^2};$$

$$\frac{3a}{a^2-b^2}=\frac{3a}{a^2-b^2}.$$

$\therefore$ $\frac{(a+b)^2}{a^2-b^2}$, $\frac{(a-b)^2}{a^2-b^2}$, and $\frac{3a}{a^2-b^2}$ are the equivalent fractions required.

Reduce to equivalent fractions with the lowest common denominator

5. $\frac{a}{bc}$, $\frac{b}{ac}$, and $\frac{c}{ab}$.

6. $\frac{a}{3c}$, $\frac{b}{4ac}$, and $\frac{c}{6a}$.

7. $\frac{3}{5x}$, $\frac{5}{2x^2}$, and $\frac{4}{10x}$.

8. $\frac{a-b}{3a}$, $\frac{a+b}{4a^2}$, and $\frac{b^2}{6a^3}$.

9. $\frac{a+5}{xy}$, $\frac{a+6}{x^2y}$, and $\frac{a+3}{xy^2}$.

10. $\frac{1}{x-1}$, $\frac{1}{x^2-1}$, and $\frac{1}{x+1}$.

11. $\frac{a-b}{a+b}$, $\frac{a+b}{a^2-b^2}$, $\frac{a^2+b^2}{a-b}$.

12. $\frac{a}{x+2}$, $\frac{a^2}{x+3}$, $\frac{2a}{x^2+5x+6}$.

13. $\frac{x^2-3}{x-3}$, $\frac{3-x^2}{x+3}$, $\frac{x^2}{x^2-9}$.

14. $\frac{a^2}{x^2-y^2}$, $\frac{b^2}{x^2+y^2}$, $\frac{a-b}{x^4-y^4}$.

260. To reduce several fractions to equivalent fractions with the lowest common denominator,

Find the L. C. M. of the denominators of the several fractions.

Divide the L. C. M. found by the denominator of each fraction, and multiply both terms of the fraction by the quotient (§ 245); or *multiply both terms of each fraction by those factors of the L. C. M. which are not contained in its denominator.*

ADDITION AND SUBTRACTION OF FRACTIONS.

261. In arithmetic *only like numbers can be added or subtracted;* and hence, to add or subtract 5 yards and 2 feet, the yards must first be reduced to feet, or the feet to the fraction of a yard; and, to add or subtract $\frac{2}{3}$ and $\frac{3}{4}$, both fractions must be changed to twelfths or to other like fractional unit.

262. The same fundamental principle holds true in algebra; and hence, to add or subtract algebraic fractions, they must first be reduced to like fractions, i.e., *to equivalent fractions with a common denominator.*

1. What is the sum and the difference of $\frac{5}{8}$ and $\frac{5}{12}$? Of $\frac{2}{3}a$ and $\frac{3}{5}a$?

The L. C. M. of 8 and 12 is 24. Hence

$\frac{5}{8}+\frac{5}{12}=\frac{15}{24}+\frac{10}{24}=\frac{25}{24}=1\frac{1}{24}$, sum.

$\frac{5}{8}-\frac{5}{12}=\frac{15}{24}-\frac{10}{24}=\frac{5}{24}$, difference.

The L. C. M. of 3 and 5 is 15. Hence

$$\frac{2a}{3}+\frac{3a}{5}=\frac{10a}{15}+\frac{9a}{15}=\frac{19a}{15}, \text{ or } \frac{19}{15}a, \text{ or } a+\frac{4a}{15}, \text{ sum.}$$

$$\frac{2a}{3}-\frac{3a}{5}=\frac{10a}{15}-\frac{9a}{15}=\frac{a}{15}, \text{ difference.}$$

2. What is (1) the sum and (2) the difference of $\frac{3x-4}{8a}$ and $\frac{4x+3}{12a}$?

The L. C. M. of $8a$ and $12a$ is $24a$. Hence

(1) $\frac{3x-4}{8a}+\frac{4x+3}{12a}=\frac{3(3x-4)+2(4x+3)}{24a}=\frac{9x-12+8x+6}{24a}=\frac{17x-6}{24a}$.

(2) $\frac{3x-4}{8a}-\frac{4x+3}{12a}=\frac{3(3x-4)-2(4x+3)}{24a}=\frac{9x-12-8x-6}{24a}=\frac{x-18}{24a}$

It must be observed that when a fraction is preceded by the sign $-$, as in (2), the *entire* numerator is to be subtracted, and hence the sign of each term must be changed or be conceived to be changed. If the numerator is inclosed in a parenthesis, as above, the signs of the included terms must be changed when the parenthesis is removed (§ 104).

263. The sum or the difference of algebraic fractions may be found *by connecting them with the proper sign, and then reducing the result to its simplest form* (§ 115).

Simplify

3. $\frac{3a}{5x}+\frac{2a}{x}$.

4. $\frac{7}{2x}+\frac{2}{x}+\frac{a}{x^2}$

5. $\frac{x^2+5}{xy}-\frac{x+3}{y}$.

6. $\frac{x}{y}-2+\frac{y}{x}$.

7. $\frac{a-x}{ax}+\frac{a+x^2}{ax^2}+\frac{a^2-x^2}{4a^2x^2}$.

8. $\frac{x}{y}-\frac{2y-x}{ab}+\frac{2y^2-xy+x^2}{aby}$.

9. $\frac{x^2y-3y^2}{5x^2}+\frac{3x^4+3y^2}{5x^2y^2}-\frac{6x^2-xy^3}{10y^2}$

10. $\frac{x+y}{x}-\frac{x+y}{y}+\frac{x^2-xy-1}{xy}$.

264. To add or subtract algebraic fractions,

Reduce the given fractions to equivalent fractions with the lowest common denominator.

Write the sum or the difference of the numerators of these fractions, as the case may be, over their lowest common denominator, and reduce the result to its simplest form.

11. Simplify $\frac{a-2}{a+1}+\frac{a-1}{a+3}-\frac{a-3}{a+2}$.

The L. C. M. is $(a+1)(a+3)(a+2)$.

$(a-2)(a+3)(a+2)=(a^2-4)(a+3)=a^3+3a^2-4a-12$;
$(a-1)(a+1)(a+2)=(a^2-1)(a+2)=a^3+2a^2-a-2$;
$(a-3)(a+3)(a+1)=-(a^2-9)(a+1)=-(a^3+a^2-9a-9)$.

Hence
$$\begin{array}{rl} & a^3+3a^2-4a-12, \text{ 1st numerator;} \\ & a^3+2a^2-a-2, \text{ 2d numerator;} \\ & -(a^3+a^2-9a-9), \text{ 3d numerator.} \\ \hline & a^3+4a^2+4a-5, \text{ sum of numerators.} \end{array}$$

$$\therefore \frac{a-2}{a+1}+\frac{a-1}{a+3}-\frac{a-3}{a+2}=\frac{a^3+4a^2+4a-5}{(a+1)(a+3)(a+2)}.$$

Since the third fraction is preceded by the sign $-$, the sign of each term in its new numerator is conceived to be changed when combining the three numerators.

Simplify

12. $\frac{1}{x+y}+\frac{1}{x-y}$.

13. $\frac{1}{a-b}-\frac{1}{a+b}$.

14. $\frac{x+y}{x-y}+\frac{x-y}{x+y}$.

15. $\frac{a}{a^3+b^3}+\frac{a}{a^3-b^3}$.

16. $\frac{y}{x-3}-\frac{3}{x-2}$.

17. $\frac{5x}{x+4}+\frac{2}{x-3}$.

18. $\frac{1+2a}{1-2a}-\frac{1-2a}{1+2a}$.

19. $\frac{x-1}{x-2}+\frac{x-3}{x^2-4}-\frac{x+1}{x+2}$.

20. $\frac{x-2}{x+3}-\frac{x+4}{x-5}$.

21. $\frac{a+bx}{a-bx}-\frac{a-bx}{a+bx}$.

22. $\frac{1}{a+b}+\frac{2b}{a^2-b^2}$.

23. $\frac{a}{a-b}-\frac{a^3}{a^3-b^3}$.

24. $\frac{1}{x-a}-\frac{1}{x+a}+\frac{2x}{x^2-a^2}$.

25. $\frac{a+b}{a-b}+\frac{a-b}{a+b}-\frac{a^2+b^2}{a^2-b^2}$.

26. $\frac{1+x^2}{1-x^2}+\frac{1-x^2}{1+x^2}-\frac{2(1+x^4)}{1-x^4}$.

27. $\frac{ax}{a-x}+\frac{x^2}{a+x}-\frac{4ax^2}{a^2-x^2}$.

28. $\frac{3x}{x+y}+\frac{x}{x-y}-\frac{2xy}{x^2-y^2}$.

29. $\frac{2a}{x^2+ax+a^2}-\frac{1}{x-a}+\frac{3a^2}{x^3-a^3}$.

30. $\frac{1}{a+b}+\frac{a-b}{a^2-ab+b^2}+\frac{ab}{a^3+b^3}$.

31. $\frac{1}{x^2+xy+y^2}-\frac{x^2}{x^4+x^2y^2+y^4}$.

32. $\frac{x}{a(x-y)}+\frac{a}{x(y-x)}$.

33. $\frac{a}{x(a-b)}-\frac{x}{a(a+b)}$.

34. $\frac{a}{a-b}-\frac{5a}{a^2-b^2}-\frac{60}{a+b}$.

35. $\frac{1}{x+y}+\frac{y}{x^2-y^2}-\frac{x}{x^2+y^2}$.

36. $\frac{3}{2a-4}+\frac{a+10}{8-2a^2}-\frac{1}{a+2}$.

37. $\frac{x}{x-1}-\frac{1}{x^2-x}+\frac{x+1}{x}$.

38. $\frac{a+b}{b}+\frac{2a}{a-b}+\frac{a^2}{ab+b^2}$.

39. $\frac{a+b}{a-b}+\frac{a-b}{a+b}-\frac{2(a^2+b^2)}{a^2-b^2}$.

40. $\frac{m^3}{(m+n)^3}-\frac{mn}{(m+n)^2}+\frac{n}{m+n}$.

41. $\frac{1}{x-3}-\frac{3}{x+3}+\frac{2x}{(x+3)^2}$.

42. $\frac{1}{x-3}+\frac{x}{x+5}-\frac{x^2+4}{x^2+2x-15}-\frac{x-3}{x+5}$.

43. $\frac{1}{x-3a}-\frac{1}{x+3a}-\frac{6a}{x^2-9a^2}+\frac{x+3a}{x-3a}$.

44. $\frac{1}{1-x}+\frac{1}{(1-x)^2}+\frac{x-1}{(1-x)^3}$.

45. $\frac{1}{x+a}+\frac{4x}{(x+a)^2}-\frac{5x}{x^2-a^2}$.

46. $\frac{1}{x^2-7x+12}-\frac{1}{x^2+x-12}$.

47. $\frac{x}{x^2-x-20}+\frac{x}{x^2-8x+15}-\frac{x}{x^2-9x+20}$.

48. $\frac{3x+2}{x^2-5x+6}-\frac{x}{x^2-8x+15}+\frac{4-x}{x^2-7x+10}$.

49. $\frac{1}{x^2+(a+b)x+ab}-\frac{1}{x^2+(a+c)x+ac}$.

265. When advantageous, the signs of both terms of a fraction may be changed (§ 247), or the signs of an *even* number of factors in either term, without changing the sign before the fraction. The sign of an *odd* number of factors in either term may be changed if the sign before the fraction be also changed.

Thus, since $(b-a)=-(a-b)$, $\frac{x}{(b-a)(c-a)}=\frac{x}{(a-b)(a-c)}$, and $\frac{x}{(b-a)(a-c)}=\frac{-x}{(a-b)(a-c)}$.

50. Simplify $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{2}{(c-a)(c-b)}$.

Change $\frac{1}{(b-a)(b-c)}$ to $-\frac{1}{(a-b)(b-c)}$, and $\frac{2}{(c-a)(c-b)}$ to $\frac{2}{(a-c)(b-c)}$, giving $\frac{1}{(a-b)(a-c)}-\frac{1}{(a-b)(b-c)}+\frac{2}{(a-c)(b-c)}$.

The L. C. M. of the new denominators is $(a-b)(a-c)(b-c)$.

$\therefore \frac{b-c-(a-c)+2(a-b)}{(a-b)(a-c)(b-c)}=\frac{a-b}{(a-b)(a-c)(b-c)}=\frac{1}{(a-c)(b-c)}$.

Simplify

51. $\frac{a}{x(a-b)} - \frac{x}{a(b-a)}$.

52. $\frac{a}{a-b} - \frac{5a}{b-a} - \frac{6b}{a+b}$.

53. $\frac{1}{(a+b)^2} - \frac{1}{b^2-a^2} - \frac{1}{(a-b)^2}$.

54. $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)}$.

55. $\frac{c}{(x-a)(a-b)} + \frac{2c}{(a-x)(b-a)}$.

56. $\frac{x^2}{(x-y)(x-z)} + \frac{y^2}{(y-x)(y-z)} + \frac{z^2}{(z-x)(z-y)}$.

57. $\frac{a^2-b}{(a-b)(a-1)} + \frac{a+b^2}{(b-a)(b+1)} - \frac{ab+1}{(a-1)(b+1)}$.

58. $\frac{x^2-yz}{(x+y)(x-z)} + \frac{y^2-xz}{(y+z)(x+y)} + \frac{z^2}{(x-z)(y+z)}$.

59. $\frac{x+c}{x^2-(a+b)x+ab} + \frac{x+b}{x^2-(a+c)x+ac} + \frac{x+a}{x^2-(b+c)x+bc}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

266. Since $\frac{a}{b} \times n = \frac{an}{b}$, and $\frac{a}{bn} \times n = \frac{a}{b}$, it follows that a fraction may be multiplied *by multiplying its numerator or by dividing its denominator.*

267. Since $\frac{an}{b} \div n = \frac{a}{b}$, and $\frac{a}{b} \div n = \frac{a}{bn}$, it follows that a fraction may be divided *by dividing its numerator or by multiplying its denominator.*

1. Multiply $\frac{3a}{b}$ by 6; $\frac{x}{my}$ by m; $\frac{3ax}{y}$ by xy.

2. Multiply $\frac{12a^2}{x^2y}$ by xy; by $4x^2$; by a^2x^2.

3. Divide $\frac{6a}{b}$ by 3; $\frac{a}{b}$ by 3; $\frac{3axy}{b}$ by ay.

4. Divide $\frac{12a^2y}{x^2}$ by $4a^2$; by $3x^2y$; by $4a^2y$.

5. Multiply $\frac{a-b}{a^2-b^2}$ by $a-b$; by $a+b$; by a^2-b^2.

6. Divide $\frac{a^2-b^2}{a-b}$ by $a-b$; by $a+b$; by a^2-b^2.

Multiplication of Fractions.

1. Multiply $\frac{a}{b}$ by $\frac{m}{n}$.

Since $\frac{m}{n} = m \div n$, $\frac{a}{b} \times \frac{m}{n} = \frac{a}{b} \times m \div n = \frac{am}{b} \div n = \frac{am}{bn}$.

$$\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}.$$

2. Multiply $\frac{12ax^2}{by^2}$ (1) by $\frac{aby}{4ax}$; (2) by $\frac{b^2y}{6x^2}$.

$$(1)\ \frac{12ax^2}{by^2} \times \frac{aby}{4ax} = \frac{12a^2bx^2y}{4abxy^2} = \frac{3ax}{y}.$$

3. Multiply $\frac{ax+bx}{3a-3b}$ (1) by $\frac{a-b}{a+b}$; (2) by $\frac{a^2-b^2}{a+b}$.

$$(2)\ \frac{ax+bx}{3a-3b} \times \frac{a^2-b^2}{a+b} = \frac{x(a+b)(a^2-b^2)}{3(a-b)(a+b)} = \frac{(a+b)x}{3}.$$

268. It is thus seen that the product of two algebraic fractions is *the product of their numerators divided by the product of their denominators.*

Multiply

4. $\frac{4a^2b}{5xy}$ by $\frac{3bx}{8ay}$.

5. $\frac{a^4b^4}{2x^2y^3}$ by $\frac{4x^4y^2}{5a^4b^2}$.

6. $\frac{3(x-y)}{5xy^2}$ by $\frac{10x^2y^3}{4(x-y)}$.

7. $\frac{x^2-y^2}{ab}$ by $\frac{a^2b}{x+y}$.

8. $\frac{a-x}{a+x}$ by $\frac{a+x}{a^2-x^2}$.

9. $\frac{a^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$.

10. $\frac{x^3-y^3}{x^3+y^3}$ by $\frac{x+y}{x-y}$.

11. $\frac{x^2-xy}{3a+3b}$ by $\frac{a+b}{x-y}$.

12. $\frac{x^2-y^2}{x+z}$ by $\frac{x+z}{x^2-2xy+y^2}$.

13. $\frac{(x+y)^2}{(x-y)^2}$ by $\frac{x-y}{x+y}$.

269. To multiply a fraction by a fraction,

Multiply the numerators together for the required numerator, and the denominators for the required denominator, and reduce the resulting fraction to its lowest terms.

Division of Fractions

270. The **reciprocal** of a fraction is 1 divided by the fraction (§ 33). Thus, the reciprocal of $\frac{5}{8}$ is $1 \div \frac{5}{8} = \frac{8}{5}$, and the reciprocal of $\frac{a}{b}$ is $1 \div \frac{a}{b} = \frac{b}{a}$. Hence the reciprocal of a fraction is *the fraction with its terms inverted.*

271. Since $\frac{m}{n} = m \div n$, $\frac{a}{b} \div \frac{m}{n} = \frac{a}{b} \div (m \div n)$; and since $\frac{a}{b} \div m$ is n times too small, $\frac{a}{b} \div (m \div n) = \left(\frac{a}{b} \div m\right) \times n = \frac{a}{bm} \times n = \frac{an}{bm}$.

It is thus seen that $\frac{a}{b}$ is divided by $\frac{m}{n}$ by multiplying $\frac{a}{b}$ by the *reciprocal* of $\frac{m}{n}$; i.e., by $\frac{n}{m}$.

1. Divide $\frac{a+b}{ab}$ by $\frac{2a}{a-b}$.

$$\frac{a+b}{ab} \div \frac{2a}{a-b} = \frac{a+b}{ab} \times \frac{a-b}{2a} = \frac{a^2-b^2}{2a^2b}.$$

2. Divide $\frac{3}{xy}$ by $\frac{12}{x^2y^2}$.

$$\frac{3}{xy} \div \frac{12}{x^2y^2} = \frac{3}{xy} \times \frac{x^2y^2}{12} = \frac{3x^2y^2}{12xy} = \frac{xy}{4}.$$

3. Divide $\frac{a^2-b^2}{a+b}$ by $\frac{a-b}{a+b}$.

$$\frac{a^2-b^2}{a+b} \div \frac{a-b}{a+b} = \frac{a^2-b^2}{a+b} \times \frac{a+b}{a-b} = a+b.$$

Divide

4. $\frac{3ab^2}{10y^2}$ by $\frac{4a^2b}{5xy}$.

5. $\frac{2b^2x^2}{5}$ by $\frac{4x^4y^2}{5a^4b^2}$.

6. $\frac{3}{2}$ by $\frac{3(x-y)}{5x^2y^2}$.

7. $\frac{a(x-y)}{3xy}$ by $\frac{x^2-xy}{xy}$.

8. $\frac{a}{a+b}$ by $\frac{a-b}{a^2+ab}$.

9. $\frac{(x+y)^2}{x-y}$ by $\frac{x+y}{(x-y)^2}$.

10. $\frac{2a(b+c)}{(b-c)^2}$ by $\frac{3a(b+c)^2}{b^2-c^2}$.

11. $\frac{x^3+y^3}{x^2-xy+y^2}$ by $\frac{x+y}{xy}$.

12. $\frac{x^3-1}{x^2+x+1}$ by $\frac{x-1}{x+1}$.

13. $\frac{a^3-b^3}{(a+b)^2}$ by $\frac{a-b}{a^2-b^2}$.

272. To divide a fraction by a fraction,

Multiply the dividend by the reciprocal of the divisor; or *multiply the dividend by the divisor with its terms inverted.*

Exercises in Multiplication and Division of Fractions.

Simplify

1. $\frac{3ax}{2b} \times \frac{4by}{9a}$.

2. $\frac{8a}{3b} \times \frac{6a^2b}{7c} \times \frac{21bc^2}{12a^2}$.

3. $\frac{3x}{2} \div \frac{5a}{7}$.

4. $\frac{3a^2x}{7y} \div \frac{6ax^2}{21y^2}$.

5. $\frac{5a^3b^2c^4}{3x^2y^2z^2} \times \frac{21x^4y^4z}{4a^2b^4c^2}$.

6. $\frac{3(a+b)}{2} \times \frac{4b}{a+b} \div \frac{a-b}{b}$.

7. $\frac{a-x}{x^2} \times \frac{x^2}{a^2-x^2} \div \frac{a}{a^2-x^2}$.

8. $\frac{x}{a+x} \times \frac{a}{a-x} \times \frac{a^2-x^2}{x^2}$.

9. $\frac{a^2+b^2}{a-b} \times \frac{a^2-b^2}{a^2} \times \frac{3a}{5}$.

10. $\frac{6(x^2-y^2)}{y} \div \frac{15(x+y)}{x-y}$.

11. $\dfrac{30(a^3-x^3)}{a^2-x^2} \div \dfrac{25(a-x)}{a+x}$.

12. $\dfrac{9x^2}{a^3+x^3} \div \dfrac{81x}{a+x}$.

13. $\dfrac{a^2-b^2}{(a-b)^2} \div \dfrac{a^2+ab}{a-b}$.

14. $\dfrac{a^3+b^3}{a^2-b^2} \times \dfrac{a-b}{a^2-ab+b^2}$.

15. $\dfrac{ab+a^2}{(a+b)^2} \div \dfrac{a^3b^2}{a^2-b^2}$.

16. $\dfrac{xy-y^2}{x(x+y)} \div \dfrac{y^2}{x(x^2-y^2)}$.

17. $\dfrac{x^2-1}{x^2+x-2} \div \dfrac{x(x+1)^2}{x+2}$.

18. $\left(1+\dfrac{a}{b}\right)\left(1-\dfrac{a}{b}\right) \div \dfrac{b}{a^2+b^2}$.

19. $\dfrac{a-b}{a^2-b^2} \times \dfrac{a-b}{a+b} \div \dfrac{(a-b)^2}{a+b}$.

20. $\dfrac{x^4-y^4}{(x-y)^2} \div \dfrac{x^2+y^2}{x^2-y^2}$.

21. $\left(\dfrac{a^3x^3}{x^3-y^3} \div \dfrac{x^2-xy+y^2}{ax}\right) \times \dfrac{x^2+xy+y^2}{a^4x^4}$.

22. $\dfrac{x^2-7x+10}{x^2+3x-88} \times \dfrac{x^2+16x+55}{x^2+2x-35} \times \dfrac{x^2-x-56}{x^2+2x-8}$.

23. $\left(\dfrac{x^2-9x+18}{x^2-17x+70} \div \dfrac{x^2-8x+15}{x^2-4x-21}\right) \times \dfrac{x^2-14x+40}{x^2-3x-18}$.

24. $\dfrac{x^2+20x+96}{x^2-15x+50} \times \dfrac{x^2-8x-20}{x^2+10x-24} \div \dfrac{x^2+10x+16}{x^2-7x+10}$.

25. $\dfrac{x^2+17x+60}{x^2-14x+49} \times \dfrac{x^2+x-56}{x^2+5x-84} \div \dfrac{x^2+13x+40}{x-7}$.

26. $\dfrac{x^3+y^3}{x^2+xy+y^2} \div \left(\dfrac{x^2-xy+y^2}{x^3-y^3} \times \dfrac{a}{x^2-y^2}\right)$.

27. $\dfrac{x(a-x)}{x^2+2ax+a^2} \times \dfrac{a(a+x)}{x^2-2ax+a^2} \times \dfrac{1}{ax}$.

28. $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}\right) \div \left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)$.

29. $\dfrac{p^3+q^3}{p^2-2pq+q^2} \times \dfrac{p^3-q^3}{p^2+2pq+q^2} \div \dfrac{p^4+p^2q^2+q^4}{p^2+q^2}$.

30. $\dfrac{x^6+y^6}{x^3-y^3} \times \dfrac{x^2+y^2}{x^4-x^2y^2+y^4} \div \dfrac{x^4+2x^2y^2+y^4}{x^4-y^4}$.

31. $\left(a^2x^4-\dfrac{1}{a^2}\right) \div \left(\dfrac{a^2x^2+1}{a^2} \times \dfrac{1}{a^2x^2-1}\right)$.

COMPLEX FRACTIONS.

273. The division of an integer by a fraction, or a fraction by an integer, or a fraction by a fraction, may be expressed by writing the dividend over the divisor, with a line between them.

Thus, $3 \div \frac{2}{5}$ may be written $\frac{3}{\frac{2}{5}}$; $\frac{2}{5} \div 3$, $\frac{\frac{2}{5}}{3}$; and $\frac{2}{3} \div \frac{3}{5}$, $\frac{\frac{2}{3}}{\frac{3}{5}}$.

274. A fraction which has a fraction in one or both of its terms is called a **complex fraction.**

275. A complex fraction may be reduced to a simple fraction *by performing the division indicated.*

Thus, $\frac{\frac{3}{5}}{\frac{5}{8}} = \frac{3}{5} \div \frac{5}{8} = \frac{3}{5} \times \frac{8}{5} = \frac{24}{25}$; and $\frac{\frac{2}{3}}{3\frac{1}{2}} = \frac{2}{3} \div \frac{7}{2} = \frac{2}{3} \times \frac{2}{7} = \frac{4}{21}$.

276. A complex fraction which contains only a simple fraction in one or both terms is best simplified *by multiplying both terms by the L. C. M. of their denominators.*

1. Reduce to a simple fraction (1) $\frac{\frac{3}{4}}{\frac{5}{8}}$; (2) $\frac{\frac{5}{6}}{6\frac{1}{4}}$.

(1) $\frac{\frac{3}{4}}{\frac{5}{8}} = \frac{\frac{3}{4} \times 8}{\frac{5}{8} \times 8} = \frac{6}{5} = 1\frac{1}{5}$. (2) $\frac{\frac{5}{6}}{6\frac{1}{4}} = \frac{\frac{5}{6} \times 12}{\frac{25}{4} \times 12} = \frac{10}{75} = \frac{2}{15}$.

2. Reduce to a simple fraction (1) $\dfrac{2 + \frac{a}{b}}{2a - \frac{b}{a}}$; (2) $1 - \dfrac{2}{1 + \frac{1}{a}}$.

(1) $\dfrac{2 + \frac{a}{b}}{2a - \frac{b}{a}} = \dfrac{\frac{2b + a}{b} \times ab}{\frac{2a^2 - b}{a} \times ab} = \dfrac{a(2b + a)}{b(2a^2 - b)}$.

(2) $1 - \dfrac{2}{1 + \frac{1}{a}} = 1 - \dfrac{\frac{2}{1} \times a}{\frac{a + 1}{a} \times a} = 1 - \dfrac{2a}{a + 1} = \dfrac{a + 1 - 2a}{a + 1} = \dfrac{1 - a}{a + 1}$.

3. Reduce to a simple fraction $\dfrac{\dfrac{a}{b}-\dfrac{a-b}{a+b}}{\dfrac{a}{b}+\dfrac{a+b}{a-b}}$.

$$\frac{\frac{a}{b}-\frac{a-b}{a+b}}{\frac{a}{b}+\frac{a+b}{a-b}}=\frac{\frac{a(a+b)-b(a-b)}{b(a+b)}\times b(a^2-b^2)}{\frac{a(a-b)+b(a+b)}{b(a-b)}\times b(a^2-b^2)}=\frac{a(a^2-b^2)-b(a-b)^2}{a(a^2-b^2)+b(a+b)^2}$$

$$=\frac{a^3-ab^2-(a^2b-2ab^2+b^3)}{a^3-ab^2+a^2b+2ab^2+b^3}=\frac{a^3-a^2b+ab^2-b^3}{a^3+a^2b+ab^2+b^3}$$

$$=\frac{(a^2+b^2)(a-b)}{(a^2+b^2)(a+b)}=\frac{a-b}{a+b}.$$

Simplify

4. $\dfrac{5\frac{3}{4}}{6\frac{2}{3}}$.

5. $\dfrac{1+\frac{x}{y}}{1+\frac{y}{x}}$.

6. $\dfrac{1-\frac{1}{x}}{x-\frac{1}{x^2}}$.

7. $\dfrac{\frac{x}{x-a}-1}{1-\frac{x}{x+a}}$.

8. $\dfrac{1}{x-\frac{1}{1+\frac{1}{a}}}$.

9. $\dfrac{c}{a-\frac{ab}{b+c}}$.

10. $\dfrac{a-\frac{1}{a}}{1+\frac{1}{a}}$.

11. $\dfrac{x-\frac{x^2}{a}}{x-\frac{x^2}{b}}$.

12. $\dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$.

13. $\dfrac{1+\frac{a}{1-a}}{\frac{1+a}{1-a}}$.

14. $\dfrac{1}{1-\frac{a}{a+b}}$.

15. $\dfrac{x-a}{1-\frac{a-x}{x+a}}$.

16. $\dfrac{\frac{1}{a-1}+\frac{1}{a+1}}{\frac{1}{b-1}+\frac{1}{b+1}}$.

17. $1+\dfrac{a}{1-\frac{2a}{1-a}}$.

18. $\dfrac{a^2-a+\frac{a-1}{a+1}}{a+\frac{1}{a+1}}$.

19. $\dfrac{\frac{x+y}{y}-\frac{y}{x+y}}{1-\frac{x}{x+y}}$.

Miscellaneous Exercises.

Reduce to simplest form

1. $\frac{(a^2+b^2)^2}{a^4-b^4}$.

2. $\frac{a^3-1}{(a-1)^2}$.

3. $\frac{a^4-a^2}{a^4-1}$.

4. $\frac{(a^2-ab)^2}{a^3-b^3}$.

5. $\frac{a^4-1}{a^6-1}$.

6. $\frac{a^4-x^4}{(a^3-x^3)(a+x)}$.

7. $\frac{(a+b)^2-(2a+b)^2}{a+b-(2a+b)}$.

8. $\frac{x^2+x-20}{x^2-x-12}$.

9. $\frac{2x^2+5x-12}{4x^2-9}$.

10. $a-\frac{a^2}{a-b}$.

11. $\frac{a^2+ax+x^2}{a^3-x^3}$.

12. $\frac{x}{x-1}-1-\frac{1}{x(x-1)}$.

13. $\frac{a}{a-1}+\frac{2a}{a+1}-\frac{2a^2}{a^2-1}$.

14. $\frac{x}{x-a}+\frac{ax}{a^2-x^2}$.

15. $\frac{1}{a-1}-\frac{2}{a}+\frac{1}{a+1}$.

16. $\frac{x+5}{y}-\frac{ax+2}{ay}$.

17. $\frac{x+2y}{x-2y}-\frac{x-2y}{x+2y}$.

18. $\frac{2}{x+1}+\frac{3x}{x^2-1}-\frac{5}{x-1}$.

19. $\frac{x}{x-y}-\frac{x^2}{x^2-y^2}+\frac{y^2}{x^2+y^2}$.

20. $\frac{3x}{4a}-\frac{6(a-x)}{5(a+x)}$.

21. $\frac{a-x}{ax}+\frac{a+x}{x^2}-\frac{a^2-x^2}{4a^2x^2}$.

22. $\frac{a^4}{x^2y^2}-\frac{a^2+x^2}{xy}\times\frac{a^2-x^2}{xy}$.

23. $\frac{a-b}{a+b}-\frac{a+b}{a-b}\times\frac{a}{a+b}$.

24. $\frac{x^2+3}{x-3}\times\frac{1}{x+3}-\frac{x-3}{x+3}$.

25. $\left(1+\frac{a-b}{a+b}\right)\div\left(1-\frac{a-b}{a+b}\right)$.

26. $\frac{a^4-b^4}{a^2-2ab+b^2}\times\left(a-\frac{2ab}{a+b}\right)$.

27. $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$.

28. $\frac{x^2-y^2}{a+b}\times\frac{a^2-b^2}{x-y}+\frac{(a-b)^2}{x^2}$.

29. $\dfrac{x^2y-3y^2}{5x^2}+\dfrac{3x^4+3y^4}{5x^2y^2}-\dfrac{6x^3-xy^3}{10y^3}$.

30. $\dfrac{x}{y}-\dfrac{2y-3x}{ab}+\dfrac{2y^2-4xy+x^2}{aby}$.

31. $\dfrac{y-z}{x}-\dfrac{x-z}{y}+\dfrac{xy^2+yz^2+x^2z}{xyz}$.

32. $\dfrac{x^2+y^2}{x-y}\div\dfrac{x^3+x^2y+xy^2+y^3}{xy}$.

33. $\dfrac{3y}{x-4}\times\dfrac{5}{x-6}\div\left(\dfrac{x}{x^2-16}\times\dfrac{y}{x-6}\right)$.

34. $\left(\dfrac{m}{1+m}+\dfrac{1-m}{m}\right)\div\left(\dfrac{m}{1+m}-\dfrac{1}{1-m^2}\right)$.

35. $\left(\dfrac{m+2n}{m+n}+\dfrac{m}{n}\right)\div\dfrac{m+2n}{n}-\dfrac{m}{m+n}$.

36. $\left(\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}\right)\div\left(\dfrac{m+n}{m-n}-\dfrac{m-n}{m+n}\right)$.

37. $\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)\div\left(\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}\right)$.

38. $\left(\dfrac{x+y}{a+b}+\dfrac{x-y}{a-b}\right)\div\left(\dfrac{x+y}{a-b}-\dfrac{x-y}{a+b}\right)$.

39. $\left(\dfrac{a-1}{x}-\dfrac{a+1}{y}\right)\div\left(\dfrac{a+1}{x}-\dfrac{a-1}{y}\right)$.

40. $\left(\dfrac{m}{m+n}+\dfrac{n}{m-n}\right)\div\left(\dfrac{m}{m+n}+\dfrac{m}{m-n}\right)$.

41. $\left(\dfrac{m}{m-n}-\dfrac{n}{m+n}\right)\div\left(\dfrac{m}{m+n}+\dfrac{n}{m-n}\right)$.

42. $\dfrac{a^4-x^4}{a^4+x^4}+\dfrac{a^6+x^6}{a^6-x^6}\times\dfrac{a^2-x^2}{a^2+x^2}\div\dfrac{a^4-a^2x^2+x^4}{a^4+a^2x^2+x^4}$.

43. $\left(\dfrac{1}{1+a}+\dfrac{a}{1-a}\right)\div\dfrac{1}{1-a}-\dfrac{a}{1+a}$.

44. $\left(\dfrac{x}{x-y}-\dfrac{x}{x+y}\right)\div\left(\dfrac{y}{x-y}-\dfrac{x}{x+y}\right)$.

CHAPTER X.

SIMPLE EQUATIONS CONTAINING FRACTIONS.

277. An equation may be cleared of fractions *by multiplying both members by the L. C. M. of the denominators* (§ 141, II.).

In practice it will often be found convenient to transpose the terms, and combine those not fractional, before clearing the equation of fractions, as shown below.

1. Solve the equation $2x - \frac{3}{5}x + 5 = \frac{3}{4}x + 12$.

Transposing and combining, $2x - \frac{3}{5}x - \frac{3}{4}x = 7$;
multiplying by 12 (L. C. M.), $24x - 8x - 9x = 84$;
combining terms, $7x = 84$;
whence $x = 12$.

2. Solve the equation $\frac{x+1}{3} - \frac{x-3}{5} - 7 = 6 - \frac{3x}{10}$.

Transposing and combining, $\frac{x+1}{3} - \frac{x-3}{5} + \frac{3x}{10} = 13$;
clearing of fractions, $10x + 10 - (6x - 18) + 9x = 390$;
transposing and combining, $13x = 362$;
whence $x = 27\frac{11}{13}$.

When the numerator of a fraction preceded by the sign $-$ contains more than one term, it will be found convenient to write the new numerator in a parenthesis, as above, and then consider the signs changed when combining the terms.

3. Solve the equation $\frac{5}{x-3} - \frac{1}{x+3} = \frac{x+30}{x^2-9}$.

The L. C. M. of the denominators is $x^2 - 9$.
Clearing equation of fractions, $5x + 15 - (x - 3) = x + 30$;
transposing and combining, $3x = 12$;
whence $x = 4$.

278. When the denominators are partly monomial and partly polynomial, it may be found advantageous to remove the monomial denominator before the polynomial.

4. Solve the equation $\frac{2x}{5} - \frac{3(x-2)}{x+1} = 3 - \frac{1-4x}{10}$.

Multiplying both members by 10, $4x - \frac{30(x-2)}{x+1} = 30 - 1 + 4x$;

transposing and combining, $\frac{-30(x-2)}{x+1} = 29$

clearing of fractions, $-30x + 60 = 29x + 29$;

transposing and combining, $-59x = -31$;

multiplying by -1, $59x = 31$;

whence $x = \frac{31}{59}$.

It may sometimes be convenient to combine the fractions in each member of the equation before clearing of fractions.

Solve the equations

5. $\frac{1}{2}x - \frac{2}{5}x = 5$.

6. $2x - 5 = \frac{3x}{4} - \frac{7}{8}$.

7. $x + \frac{1}{2}x + \frac{1}{8}x = 11$.

8. $x - \frac{x}{3} = \frac{x}{4} + 15$.

9. $\frac{x}{2} + \frac{x}{3} = 13 - \frac{x}{4}$.

10. $x + \frac{x}{2} - \frac{x}{3} = 4x - 17$.

11. $5 - \frac{x+4}{11} = x - 3$.

12. $\frac{2x+3}{3} - \frac{x+3}{4} = \frac{x-4}{5} + 3$.

13. $5x - \frac{5x}{2} + 12 = \frac{4x}{3} + 26$.

14. $\frac{x-5}{4} + 6x = \frac{284 - x}{5}$.

15. $x + \frac{1}{3}(11 - x) = \frac{1}{2}(19 - x)$.

16. $\frac{4}{x-2} = \frac{3}{x-4}$.

17. $\frac{3}{2x+4} = \frac{2}{3x-4}$.

18. $\frac{5}{2x} + \frac{3}{x} = \frac{11}{12}$.

19. $x - \frac{26-x}{2} = \frac{x-11}{3}$.

20. $\frac{11}{3x} - \frac{1}{6} = \frac{7}{2x}$.

21. $\frac{x^2+5}{x-3} - (x-7) = 3$.

22. $\frac{x-1}{x+1} = \frac{6x-6}{5x+5}$.

23. $\frac{12x+97}{3x+1} = \frac{4x+16}{x-2}$.

24. $\frac{3x^2-5x+2}{x-2}=3x+10.$

25. $\frac{2x-5}{3x-7}=\frac{2x-7}{3x-5}.$

26. $\frac{3-2x}{x-5}=\frac{2x-7}{4-x}.$

27. $\frac{x+1}{x-1}-\frac{x-3}{x+3}=\frac{8}{x}.$

28. $\frac{5(x-2)}{x+2}-\frac{2(x-3)}{x+3}=3.$

29. $\frac{4}{x^2-1}+\frac{1}{x+1}=\frac{1}{1-x}.$

30. $\frac{3}{1-x}-\frac{2}{1+x}-\frac{1}{1-x^2}=0.$

31. $\frac{3}{x^2-9}+\frac{1}{x+3}=\frac{2}{3-x}.$

32. $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}.$

33. $\frac{x-1}{x-2}+\frac{x+1}{x+2}=\frac{2(x-2)}{x+2}.$

34. $\frac{9-2y}{3}-\frac{5y+8}{6}+10=\frac{3y}{5}.$

35. $3x-\frac{1}{5}(2x+6)=-1-\frac{1}{2}(11x-32).$

36. $2-\frac{1}{3}(6x-4)=\frac{1}{3}(4x-18)-x.$

37. $\frac{x+4}{4}-7=\frac{x-23}{5}-\frac{x-1}{7}.$

38. $\frac{x-4}{4}-3x+4=\frac{1}{12}-\frac{5x+14}{3}.$

39. $\frac{1}{3}(7x+5)-\frac{1}{5}(16+4x)+6=\frac{1}{2}(3x+9).$

40. $\frac{1}{5}(3x+4)+\frac{1}{4}(16-x)+\frac{1}{2}(3-7x)=0.$

41. $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7}{3}(x+2).$

42. $\frac{3}{3x+2}=\frac{1}{2x-4}-\frac{1}{3-2x}.$

43. $\frac{x+1}{x-1}-\frac{x}{x-2}=\frac{9-x}{7-x}+\frac{x-8}{6-x}.$

44. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x.$

45. $2x^3-\frac{x^3}{2}+x^3=5x^3-2x^3.$

46. $\frac{(3x-2)(2x-3)}{6}-\frac{6x-8}{5}+\frac{2}{15}=x^2-\frac{x+10}{3}.$

LITERAL EQUATIONS.

279. The known numbers in an equation may be represented by *letters*, the first letters of the alphabet being commonly used for this purpose (§ 11). Thus, in the equation $ax + bx = ab$, a and b may represent known numbers; i.e., numbers regarded as known.

280. An equation in which some or all of the known numbers are represented by letters is called a **literal equation.**

1. Solve the equation $\frac{a}{x-b} = \frac{b}{x-a}$.

Clearing of fractions, $ax - a^2 = bx - b^2$;
transposing terms, $ax - bx = a^2 - b^2$;
factoring, $(a-b)x = (a-b)(a+b)$;
dividing by $a-b$, $x = a+b$.

2. Solve the equation $\frac{x}{a} + b = \frac{x}{c} + d$.

Transposing terms, $\frac{x}{a} - \frac{x}{c} = d - b$;
clearing of fractions, $cx - ax = acd - abc$;
factoring, $(c-a)x = ac(d-b)$;
dividing by $c-a$, $x = \frac{ac(d-b)}{c-a}$.

Solve

3. $ax + c = bx + 2c$.
4. $\frac{1}{2}ax + \frac{1}{3}bx = c$.
5. $\frac{ax}{b} - c = \frac{mn}{n} + d$.
6. $\frac{x}{a} - \frac{x-b}{c} = \frac{d}{c}$.
7. $\frac{x}{a+b} - \frac{x}{a-b} = -1$.
8. $\frac{x}{m} + n = \frac{x}{a} - b$.
9. $\frac{a-b}{a+b}x + c = \frac{a+b}{a-b}x - d$.
10. $\frac{m-x}{x+p} = \frac{x+r}{n-x}$.
11. $\frac{a^2 - ax}{b} - \frac{b^2 + bx}{a} = x$.

12. $\frac{ax-a^2}{b}+\frac{bx-b^2}{a}=x.$

13. $\frac{ax-b}{x+3}-\frac{ax+b}{x-3}=\frac{5}{x^2-9}.$

14. $\frac{a}{x-a}-\frac{b}{x+a}=\frac{b}{x^2-a^2}.$

15. $\frac{a}{x}+\frac{b}{x}-\frac{d}{x}=\frac{ax+b}{x^2}.$

16. $\frac{a^2}{x}-ab=\frac{1}{x}+b.$

17. $\frac{a-b}{x-b}=\frac{a+b}{x+b}-\frac{ax}{x^2-b^2}.$

18. $(x+a)(x+b)=(x+c)^2.$

19. $(x-a)(x-b)=(x-a-b)^2.$

20. $m^2x+2mn-n^2x=m^2+n^2.$

21. $6n^2x-2m^2=3mn-4mnx.$

22. $\frac{x}{p}+\frac{x}{q}+\frac{x}{r}=pq+pr+qr.$

PROBLEMS.

1. Divide the number 132 into two parts such that one part may be $\frac{5}{6}$ of the other.

2. Divide the number a into two parts such that one part may be $\frac{m}{n}$ of the other.

3. Divide the number 108 into three parts such that $\frac{1}{2}$ of the first equals $\frac{1}{3}$ of the second, and also $\frac{1}{4}$ of the third.

4. Divide the number n into three parts such that $\frac{1}{a}$ of the first, $\frac{1}{b}$ of the second, and $\frac{1}{c}$ of the third, will be equal.

Let x = first part.

Then $\frac{x}{a}\times b=\frac{bx}{a}$ = second part,

and $\frac{x}{a}\times c=\frac{cx}{a}$ = third part.

Hence $x+\frac{bx}{a}+\frac{cx}{a}=n.$

Solving the equation, $x=\frac{an}{a+b+c}$, first part.

Whence $\frac{bx}{a}=\frac{bn}{a+b+c}$, second part;

$\frac{cx}{a}=\frac{cn}{a+b+c}$, third part.

5. Divide the number m into three parts such that $\frac{1}{a}$ of the first, $\frac{1}{2a}$ of the second, and $\frac{1}{3a}$ of the third, will be equal.

6. A man walked 87 miles in 3 days, and $\frac{1}{8}$ of the distance walked the first day equaled $\frac{1}{4}$ of the distance walked the second day, and $\frac{1}{3}$ of the distance walked the second day equaled $\frac{1}{2}$ the distance walked the third day. How far did he walk each day?

7. A father bequeathed $\frac{5}{12}$ of his estate to his eldest son, $\frac{4}{7}$ of the remainder to his second son, and the rest to his youngest son; and the eldest son received $1200 more than the younger. What was the share of each?

8. Divide $735 among three persons so that the second will have $\frac{3}{4}$ as much as the first, and the third $\frac{1}{2}$ as much as the other two together.

9. A man bought a horse and carriage for $275, and $\frac{1}{8}$ of the cost of the carriage plus $33 was equal to $\frac{1}{3}$ of the cost of the horse. What was the cost of each?

10. A man bought a horse, saddle, and bridle for $150. The cost of the saddle was $\frac{1}{6}$ of the cost of the horse, and the cost of the bridle was $\frac{1}{2}$ the cost of the saddle. What was the cost of each?

11. Ten years ago A's age was $\frac{3}{4}$ of B's age, and 10 years hence A's age will be $\frac{5}{6}$ of B's age. What is the age of each now?

12. At the time of marriage a wife's age was $\frac{3}{5}$ of the age of her husband, and 12 years after marriage her age was $\frac{5}{7}$ of her husband's age. How old was each at marriage?

13. A man can do $\frac{1}{5}$ of a piece of work in a day, and a boy can do $\frac{1}{8}$ of it in a day. In how many days can both of them working together do it?

14. A can do a piece of work in a days, and B in b days. In how many days can both together do it?

Let x = number of days.

Then $\frac{1}{x}$ = part both can do in one day;

$\frac{1}{a}$ = part A can do in one day;

$\frac{1}{b}$ = part B can do in one day.

Hence $\frac{1}{a}+\frac{1}{b}=\frac{1}{x}$

Clearing of fractions, $bx + ax = ab$;

whence $x=\frac{ab}{a+b}$, number of days.

15. A can do a piece of work in 8 days, and B in 12 days. In how many days can both together do it?

16. A can do a piece of work in 3 days, B in 5 days, and C in 6 days. In how many days can they together do it?

17. A and B can do a piece of work in 8 days, A and C in 10 days, and B and C in 12 days. In how many days can A, B, and C together do it? In how many days can each alone do it?

18. A and B working together can build a wall in 8 days, and A alone can build it in 12 days. In how many days can B alone build it?

19. A and B can build a fence in 8 days, and with C's help they can build it in 6 days. How long will it take C alone to build the fence?

20. A man spent $\frac{3}{4}$ of his money, and then earned $\frac{1}{2}$ as much as he had spent, and then had \$ 21 less than he had at first. How much money did he have at first?

21. An estate was so divided between two heirs that $\frac{2}{3}$ of the share of the first equaled $\frac{3}{4}$ of the share of the second; and the difference of their shares was \$ 362. What was the share of each?

22. A man paid $ 8100 for two farms, and $\frac{4}{5}$ of the cost of the larger farm was equal to $\frac{9}{10}$ of the cost of the smaller. What was the cost of each?

23. A piece of carpeting containing 135 yards was cut into three carpets such that $\frac{3}{10}$ of the number of yards in the first carpet equaled $\frac{1}{6}$ of the number of yards in the second, and $\frac{2}{5}$ of the number of yards in the third. How many yards were in each carpet?

24. A cistern can be filled in 4 hours by two pipes running together, and in $6\frac{1}{4}$ hours by one pipe alone. In how many hours can the other pipe alone fill it?

25. Two stoves consumed a certain amount of coal in 12 days, and the smaller stove would consume it in 30 days. In how many days would the larger stove consume it?

26. A farmer paid $ 410 for sheep of different grades. For $\frac{1}{5}$ of the whole number he paid $ 10 each; for $\frac{1}{3}$ of the whole, $ 7.50 each; and for the rest, $ 5 each. How many sheep did he buy?

27. The sum of $ 100.50 is contributed by 100 persons. Some give 50¢ each; $\frac{1}{2}$ as many, 75¢ each; and the rest, $ 1.50 each. How many contributors of each class?

28. A and B have equal incomes. A lays up $\frac{1}{5}$ of his each year; but B spends $\frac{1}{2}$ more than A, and in 3 years finds himself $ 600 in debt. What is the income of each?

29. The perimeter of a triangular field is 81 rods, and the longest side is $\frac{1}{8}$ longer than one of the other sides, and twice as long as the other. What is the length of each side?

30. A woman sold from her basket $\frac{1}{2}$ the number of eggs in it, and then sold $\frac{1}{3}$ of the eggs remaining, and then had 20 eggs left. How many eggs did she sell?

31. The difference of the squares of two consecutive numbers is 21. What are the numbers?

32. A factory employs 200 men, 150 women, and 80 children, and they receive each week as wages \$ 4110, 3 men receiving as much as 4 women or 8 children. How much did each man, woman, and child receive per week?

33. Divide \$ 6000 among A, B, and C, giving A \$ 300 more than B, and C one half as much as A and B together.

34. Divide m dollars among A, B, and C, giving A n dollars more than B, and n dollars less than C.

35. A merchant bought a lot of velvet at \$ 1.50 a yard, and then sold one half of it at \$ 2 a yard, one third of it at \$ 1.75 a yard, and the remainder at \$ 1.25 a yard, and made a profit of \$ 157.50. How many yards did he buy?

36. Divide m dollars between A and B in the ratio of a to b.

Let $ax =$ A's share,

and $bx =$ B's share.

Then $ax + bx = m$;

whence $x = \frac{m}{a+b}$.

Hence $ax = \frac{am}{a+b}$, A's share;

$bx = \frac{bm}{a+b}$, B's share.

37. Divide \$ 120 between two men in the ratio of 3 to 5.

SUGGESTION. This problem may be solved by substituting \$ 120 for m, 3 for a, and 5 for b, in the foregoing formulas, for A's and B's shares.

38. The difference between two numbers is 25, and the greater is to the less as 3 to 2. What are the numbers?

39. The difference between two numbers is m, and the greater is to the less as a to b. What are the numbers?

40. A boy bought apples at 2 for a cent, and as many more at 3 for a cent, and then sold them at the rate of 5 for 3 cents, and gained 11 cents. How many apples did he buy?

41. A fruit vender bought a certain number of pears at 2 cents each, $\frac{7}{10}$ as many lemons at 3 cents each, and $\frac{1}{5}$ as many oranges at 4 cents each, and paid \$ 1.96 for the lot. How many of each did he buy?

42. A party of 20 persons pay \$ 40 for their railroad tickets. The full fare is \$ 2.50, but the children are charged only half fare. How many children in the party?

43. A miller made a mixture of barley, corn, and oats, using 3 bushels of barley to 4 of corn and 5 of oats. How many bushels of each grain did he use in a mixture of 72 bushels?

44. When a colonel tried to draw up his regiment in a solid square with a certain number of men in the front rank, he had 35 men too many, and when he put one man more in the front rank he had 30 men too few. How many men in the regiment?

SUGGESTION. Let x = the number of men in the front rank of the *first* solid square.

45. A woman bought a dollar's worth of postage stamps, receiving a certain number of five-cent stamps, twice as many two-cent stamps less 3, and 3 times as many one-cent stamps less 2. How many stamps of each kind did she buy?

46. A speculator bought a piece of land at \$ 450 an acre, reserved 5 acres for himself, and laid out the rest in building lots which he sold at \$ 1000 an acre, gaining in the transaction \$ 11,500 besides the 5 acres reserved. How many acres of land did he buy?

47. A man engaged to work for 30 days on the conditions that he was to receive \$ 1.50 for each day he worked, and forfeit 50¢ for each idle day. At the end of the 30 days he received \$ 27. How many days had he worked?

48. A man engaged to work for m days on the conditions that he was to receive a dollars for each day he worked, and forfeit b dollars for each idle day. At the end of m days he received n dollars. How many days had he worked?

49. A farmer can mow a field in 12 hours, his oldest son in 16 hours, and his second son in 18 hours. In how many hours can the three together mow it?

50. A, B, and C can do a piece of work in 20 days; A and B can do it in 40 days; and A and C in 30 days. In how many days can each alone do it?

51. A man being asked his age replied that $\frac{3}{5}$ of his age 10 years ago is equal to $\frac{2}{5}$ of his age 10 years hence. What was his age?

52. At what time between 2 and 3 o'clock are the hands of a watch together? At what time between 4 and 5?

53. At what time between 3 and 4 o'clock are the hands of a watch opposite each other? At what time between 8 and 9 o'clock?

54. A man spends one fifth of his yearly income for house rent, one half of the remainder for provisions, two fifths of the remainder for other expenses, and lays up $\$ 240$. What is his yearly income?

55. A steamer, running 18 miles an hour, follows a ship 16 miles off, that is sailing 10 miles an hour. How many miles must the steamer run to overtake the ship?

56. A courier starts from a certain place and travels at the rate of $10\frac{3}{5}$ miles an hour. Two hours later a second courier, traveling $13\frac{3}{5}$ miles an hour, is sent to recall the first. In how many hours will the second courier overtake the first?

57. A person has just 5 hours at his disposal. How far can he ride with a friend in a buggy, going 10 miles an hour, and walk back at the rate of 4 miles an hour?

GENERAL PROBLEMS.

281. When the given numbers in a problem are represented by letters, it is called a **general problem**; and its solution is a general solution for all problems of that class. It also serves as a model solution. Several of the foregoing problems are general problems.

282. The result obtained by the solution of a general problem is called a **solution formula**, or, briefly, a **formula**; and, by substituting for the letters in such formula the particular numbers given in a similar problem, the numerical answer to such problem is obtained.

1. The time past noon is n times the time to midnight. What is the time of day?

Let x = time past noon;
then $12 - x$ = time to midnight.
Hence $x = n(12 - x)$.
Transposing terms, $x + nx = 12n$;
whence $x = \frac{12n}{1+n}$, time past noon; (1)

$12 - x = 12 - \frac{12n}{1+n} = \frac{12}{1+n}$, time to midnight. (2)

2. The time past noon is $\frac{2}{3}$ of the time to midnight. What is the time of day?

Let x = time past noon, and substitute $\frac{2}{3}$ for n in Formulas (1) and (2) above, and we have

$$x = \frac{12n}{1+n} = \frac{12 \times \frac{2}{3}}{\frac{5}{3}} = \frac{24}{5} = 4\frac{4}{5} = 4 \text{ h. } 48 \text{ m., time past noon.}$$

$$12 - x = \frac{12}{1+\frac{2}{3}} = \frac{12}{\frac{5}{3}} = 7\frac{1}{5} = 7 \text{ h. } 12 \text{ m., time to midnight.}$$

3. A man, being asked the time of day, said that $\frac{3}{4}$ of the time past noon equals $\frac{3}{5}$ of the time to midnight. What was the hour of day?

4. What is the time of day when $\frac{2}{3}$ of the time to noon is equal to $\frac{3}{5}$ of the time past midnight?

5. A courier pursues a second courier, who has a start of m miles; and the first courier travels at the rate of a miles an hour, and the second at the rate of b miles an hour. In how many hours will the first courier overtake the second?

Let x = the number of hours;
then ax = distance traveled by first courier,
and bx = distance traveled by second courier.
Hence $ax - bx = m$;
whence $x = \frac{m}{a-b}$, number of hours.

6. An express train running 36 miles an hour follows an accommodation train running 24 miles an hour, and the accommodation train has a start of 84 miles. In how many hours will the express train overtake the accommodation train?

Let x = number of hours.

Since $x = \frac{m}{a-b}$, and $m = 84$, $a = 36$, and $b = 24$,

$$x = \frac{84}{36-24} = 7, \text{ number of hours.}$$

7. A train sets out from A for B, which is n miles distant, running a miles an hour; c hours later another train leaves B for A, running b miles an hour. How far will each train have run when it meets the other?

Let x = the number of hours run by first train;
then ax = distance run by first train;
$x - c$ = number of hours run by second train;
$b(x-c)$ = distance run by second train.
Hence $ax + b(x-c) = n$;
whence $x = \frac{n+bc}{a+b}$;

$$ax = \frac{a(n+bc)}{a+b}, \text{ distance run by first train;} \quad (1)$$

$$b(x-c) = \frac{b(n-ac)}{a+b}, \text{ distance run by second train.} \quad (2)$$

8. The railroad distance from Buffalo to Chicago is 540 miles. An express train leaves Buffalo for Chicago, running at the rate of 35 miles an hour; and $5\frac{1}{4}$ hours later an express train leaves Chicago for Buffalo, running at the rate of 40 miles an hour. How far will each train have run when they meet?

SUGGESTION. Solve this problem by the use of the general formulas (1) and (2) obtained in solving Problem 7.

9. Divide the number m into four parts such that the first increased by n, the second diminished by n, the third multiplied by n, and the fourth divided by n, will be equal.

Let x = number to which the results are equal;

then $x - n$ = first part;

$x + n$ = second part;

$\frac{x}{n}$ = third part;

nx = fourth part.

Hence $x - n + x + n + \frac{x}{n} + nx = m.$

Combining terms, $2x + \frac{x}{n} + nx = m;$

clearing of fractions, $2nx + x + n^2x = mn;$

factoring, $(n^2 + 2n + 1)x = mn;$

whence $x = \frac{mn}{(n+1)^2}.$

$x - n = \frac{mn}{(n+1)^2} - n$, first part;

$x + n = \frac{mn}{(n+1)^2} + n$, second part;

$\frac{x}{n} = \frac{m}{(n+1)^2}$, third part;

$nx = \frac{mn^2}{(n+1)^2}$, fourth part.

10. Divide the number 80 into four parts such that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, will all be equal.

SUGGESTION. Let x denote the number to which the several results are equal, $x - 3$ denoting the first part, $x + 3$ the second part, and so on; or solve by substituting in the foregoing formulas.

11. Divide the number 100 into three parts such that the first increased by 8, the second diminished by 8, and the third divided by 3, will be equal.

12. Divide the number m into three parts such that the first increased by n, the second diminished by n, and the third divided by n, will be equal.

13. The sum of two numbers is m, and their difference is n. What are the numbers?

Let $x =$ the larger number;

then $x - n =$ the smaller number.

Hence $x + x - n = m$.

Transposing terms, $2x = m + n$;

whence $$x = \frac{m+n}{2}, \text{ the larger number.} \quad (1)$$

$$x - n = \frac{m+n}{2} - n = \frac{m-n}{2}, \text{ smaller number.} \quad (2)$$

283. It is seen from these two formulas that when the sum and the difference of two numbers are given,

I. To find the larger number, *add the sum and the difference of the two numbers, and divide the result by 2.*

II. To find the smaller number, *subtract the difference of the two numbers from their sum, and divide the result by 2.*

284. In like manner the formula obtained by the solution of any general problem may be expanded into a *rule* for the solution of all like problems.

Such a formula may also be used for the solution of problems that vary somewhat from the general problem from which it is derived.

14. Expand into a rule the formula reached by the solution of Problem 1.

15. Expand into a rule the formula reached in the solution of Problem 5.

285. Percentage formulas. — The problems in simple percentage involve three numbers such that, if two are given, the third may be found. These numbers may be represented by letters, as follows:

b = *base*, or the number of which the per cent is taken.

r = *rate per cent*, or the number of hundredths.

p = *percentage*, or the number found by taking the given per cent of the base.

Since $p = br$, $r = \frac{p}{b}$, and $b = \frac{p}{r}$, and hence the formulas:

(1) $p = br$. (2) $r = \frac{p}{b}$. (3) $b = \frac{p}{r}$. (4) $b = \frac{b+p}{1+r}$.

By these formulas all the problems in simple percentage may be solved.

16. If 5% of a certain ore is silver, how much silver is there in 3740 pounds of the ore?

17. 2400 pounds of iron ore yielded 1008 pounds of iron. What per cent of the ore was iron?

18. If copper ore is 25% copper, how many pounds of the ore will yield 1000 pounds of copper?

286. Interest formulas. — The problems in interest involve five numbers or elements: the *principal*, the *rate*, the *time*, the *interest*, and the *amount*. These five numbers are so related, that, if any three are given, the other two may be found.

287. These numbers may be represented by letters, as follows:

p = *principal*, or the money on which interest is computed.

r = *rate per cent*, or the number of hundredths.

t = *time*, expressed in years or parts of a year.

i = *interest*, or the money paid for the use of the principal.

a = *amount*, or the sum of principal and interest.

288. The interest on any sum of money for one year is found by multiplying the principal by the rate per cent, ex-

pressed as hundredths; and the interest for any given time is found by multiplying the interest for one year by the number of years. Hence $i = prt$.

Since $i = prt$, $r = \frac{i}{pt}$, $t = \frac{i}{pr}$, and $p = \frac{i}{rt}$; and we thus obtain the following special formulas for the solution of the several classes of problems in interest:

(1) $i = prt$. (2) $r = \frac{i}{pt}$. (3) $t = \frac{i}{pr}$. (4) $p = \frac{i}{rt}$.

19. What is the interest of \$ 160.80 for 2 yr. 3 mo. at 8 %?

20. The interest of \$ 95.40 for 3 yr. 9 mo. is \$ 28.62. What is the rate per cent?

21. The interest of \$ 56.78 for a certain time at 10 % was \$ 22.24. What was the time?

22. What principal will produce \$ 86.80 of interest in 2 yr. 4 mo. at 6 %?

289. Since the amount is the sum of principal and interest ($a = p + i$), $a = p + ptr$; and hence, factoring, $a = p(1 + rt)$, and $p = \frac{a}{1 + rt}$. Again, since $a = p + ptr$, $a - p = ptr$; and hence $r = \frac{a - p}{pt}$, and $t = \frac{a - p}{pr}$.

We thus obtain the following formulas:

(1) $a = p + i = p + prt$. (3) $t = \frac{a - p}{pr}$.

(2) $r = \frac{a - p}{pt}$. (4) $p = a - i = \frac{a}{1 + rt}$.

23. A note of \$ 95.40 at 8 %, when paid, amounted to \$ 124.02. How long did the note run?

24. A note bearing interest at 9 % amounted in 3 yr. 2 mo. to \$ 360.75. What was the face of the note?

NOTE. For additional problems in percentage and interest, see White's "New Complete Arithmetic," pp. 178–192 and 240–246.

CHAPTER XI.

SIMULTANEOUS EQUATIONS.

SIMPLE EQUATIONS WITH TWO UNKNOWN NUMBERS.

290. In a simple equation with only one unknown number, there is one value of this number, and only one, that will satisfy the equation. Such an equation is said to be **determinate**; i.e., the unknown number in it has a definite value. Thus, (1) $3x = 12$, and (2) $\frac{2}{3}x = 10$, are determinate equations, the value of x in (1) being 4, and in (2), 15.

291. When an equation contains two unknown numbers, as $x + y = 12$, an indefinite number of values of these unknown numbers will satisfy the equation, and for this reason the equation is said to be **indeterminate**. Thus, in $x + y = 12$, the value of x will vary with the value of y, and the value of y will vary with the value of x. If $y = 4$, $x = 8$; if $y = 2$, $x = 10$; if $x = 7$, $y = 5$; if $x = 15$, $y = -3$; and so on.

292. Two equations that express *different relations* between two unknown numbers are said to be **independent**. Thus, $x + y = 12$, and $xy = 35$, express different relations between x and y, and hence are independent equations.

Independent equations cannot be made to assume the same form. Thus, $x + y = 12$ (1), and $2x + 2y = 24$ (2), can be made to assume the same form by multiplying each member of (1) by 2, or dividing each member of (2) by 2, and hence these equations are not independent.

293. The unknown numbers in two independent equations may have the same value in both; and, when this is the case, the substitution of these values for the unknown numbers will satisfy both equations *simultaneously*. Thus, the values of x

and y in $x - y = 2$, and $xy = 35$, are respectively 7 and 5, and the substitution of these values satisfies both equations.

294. Two independent equations in which the two unknown numbers have each the same value are called **simultaneous equations.** Thus, $x + y = 13$, and $x - y = 3$, are simultaneous equations.

Elimination.

295. The solution of two simultaneous equations involves their combination in such manner as to remove or eliminate one of the unknown numbers.

Thus, let $2x + y = 42$, and $3x - y = 33$, be two simultaneous equations. Adding the first members and the second members of the two equations (Ax. 1), we have

$$\begin{aligned} 2x + y &= 42 \quad (1)\\ 3x - y &= 33 \quad (2)\\ \hline 5x \quad &= 75 \quad \therefore x = 15. \end{aligned}$$

Substituting 15 for x in (1), we have $30 + y = 42$. $\therefore y = 12$.

296. The process of combining two simultaneous equations in such manner as to obtain a single equation with only one unknown number, is called **elimination.**

297. There are three general methods of elimination: to wit,

I. *By addition or subtraction.*
II. *By substitution.*
III. *By comparison*

298. I. Elimination by addition or subtraction.

1. Find the values of x and y in the equations

$$\begin{cases} 5x + 4y = 40, & (1)\\ 7x - 2y = 18. & (2) \end{cases}$$

Multiplying both members of (2) by 2, $14x - 4y = 36$; (3)
adding (1) and (3) member to member, $19x = 76$;
whence $x = 4$.

Substituting 4 for x in (1), $20 - 4y = 40$;

transposing, etc., $4y = 20$;

whence $y = 5$.

Hence $x = 4$, and $y = 5$; and these values of x and y will satisfy the given equations.

2. Solve the equations
$$\begin{cases} 4x + 3y = 55, & (1) \\ 5x + 2y = 60. & (2) \end{cases}$$

Multiplying (1) by 2, and (2) by 3, $8x + 6y = 110$, (3)

$15x + 6y = 180$; (4)

subtracting (3) from (4), $7x = 70$;

whence $x = 10$.

Substituting 10 for x in (1), $40 + 3y = 55$;

transposing, $3y = 15$;

whence $y = 5$.

3. Solve the equations
$$\begin{cases} \frac{x}{4} + \frac{y}{5} = 8, & (1) \\ \frac{3x}{5} + \frac{2y}{3} = 22. & (2) \end{cases}$$

Clearing (1) and (2) of fractions, $5x + 4y = 160$, (3)

$9x + 10y = 330$; (4)

multiplying (3) by 5, and (4) by 2, $25x + 20y = 800$, (5)

$18x + 20y = 660$; (6)

subtracting (6) from (5), $7x = 140$;

whence $x = 20$.

Substituting 20 for x in (3), $100 + 4y = 160$;

transposing, $4y = 60$;

whence $y = 15$.

It is sometimes not necessary to clear the equations of fractions, since the coefficients of one of the letters may otherwise be made the same, as shown in the following solution.

4. Solve the equations

$$\begin{cases} \frac{5x}{8}+\frac{y}{6}=6, & (1) \\ \frac{3x}{4}-\frac{y}{3}=4. & (2) \end{cases}$$

Multiplying (1) by 2, $\frac{5x}{4}+\frac{y}{3}=12;$ (3)

adding (2) and (3), $\frac{8x}{4}$ or $2x=16;$

whence $x=8.$

Substituting 8 for x in (1), $5+\frac{y}{6}=6;$

transposing, $\frac{y}{6}=1.$ $\therefore y=6.$

Solve the equations

5. $\begin{cases} x+y=18, \\ 2x-y=21. \end{cases}$

6. $\begin{cases} x+2y=20, \\ x-3y=-10. \end{cases}$

7. $\begin{cases} 4x-3y=13, \\ 6x-4y=22. \end{cases}$

8. $\begin{cases} 7x+4y=94, \\ 5x-2y=38. \end{cases}$

9. $\begin{cases} x+3y=36, \\ x-2y=16. \end{cases}$

10. $\begin{cases} 5x-4y=35, \\ 7x-3y=62. \end{cases}$

11. $\begin{cases} 3x+4y=44, \\ 5y-3x=1. \end{cases}$

12. $\begin{cases} 3x+\frac{2y}{3}=28, \\ 5x-\frac{y}{2}=37. \end{cases}$

13. $\begin{cases} \frac{x}{4}+\frac{y}{3}=9, \\ \frac{x}{3}-\frac{y}{6}=1. \end{cases}$

14. $\begin{cases} \frac{2x}{5}-\frac{3y}{4}=5, \\ \frac{3x}{8}-\frac{3y}{5}=6\frac{3}{4}. \end{cases}$

15. $\begin{cases} \frac{3x}{4}+\frac{y}{6}=14, \\ \frac{x}{8}+\frac{y}{6}=4. \end{cases}$

16. $\begin{cases} \frac{x+y}{5}+2y=5, \\ 3x-\frac{x-y}{2}=8\frac{1}{2}. \end{cases}$

To eliminate by addition or subtraction,

Multiply or divide the given equations by such numbers as will make the coefficients of one of the unknown numbers equal in the resulting equations.

If the equal coefficients have unlike signs, add the resulting equations; and, if they have like signs, subtract one equation from the other.

NOTE. The adding of the first members and the second members of two equations is called briefly *the adding of the equations;* and in like manner the subtracting of the first and the second members of one equation respectively from the first and the second members of another equation is called briefly *the subtracting of one equation from the other.*

299. II. Elimination by substitution.

17. Solve the equations $\begin{cases} x+2y=20, & (1) \\ 2x-3y=5. & (2) \end{cases}$

Eliminate x.

From (1) we find $x = 20 - 2y.$

Substituting $20-2y$ for x in (2), $2(20-2y)-3y=5;$

simplifying, $40-7y=5;$

transposing, $-7y=-35;$

dividing by -7, $y=5.$

Substituting 5 for y in (1), $x+10=20;$

whence $x=10.$

18. Solve the equations $\begin{cases} 5y-3x=-1, & (1) \\ 3x+4y=37. & (2) \end{cases}$

Eliminate y.

From (1) we find $y=\frac{3x-1}{5}.$

Substituting $\frac{3x-1}{5}$ for y in (2), $3x+\frac{12x-4}{5}=37;$

clearing of fractions, etc., $27x=189;$

whence $x=7.$

Substituting 7 for x in (1), $5y-21=-1;$

whence $y=4.$

Solve the equations

19. $\begin{cases} x+4y=21, \\ 2x+3y=22. \end{cases}$

20. $\begin{cases} 2x+3y=27, \\ 4x-\ y=\ 5. \end{cases}$

21. $\begin{cases} 2x+5y=19, \\ x-7y=\ 0. \end{cases}$

22. $\begin{cases} 5x+2y=29, \\ 2y-\ x=-1. \end{cases}$

23. $\begin{cases} y-2z=-6, \\ y+2z=34. \end{cases}$

24. $\begin{cases} 5x-3y=34, \\ 5y-3x=\ 2. \end{cases}$

25. $\begin{cases} 4x+3y=22, \\ 2y-4x=-12. \end{cases}$

26. $\begin{cases} 3x-\frac{2z}{3}=10, \\ 2x-\frac{3z}{4}=\ 3. \end{cases}$

27. $\begin{cases} \frac{x}{2}+\frac{y}{3}=8, \\ \frac{x}{3}+\frac{y}{2}=7\frac{5}{6}. \end{cases}$

28. $\begin{cases} \frac{x-y}{2}+\frac{x+y}{2}=6, \\ 2x-3y=0. \end{cases}$

To eliminate by substitution,

Find from one of the given equations the value of the unknown number to be eliminated.

Substitute this value for the unknown number in the other equation, and solve the resulting equation.

300. III. Elimination by comparison.

29. Find the values of x and y in

$$\begin{cases} x+2y=25, & (1) \\ 2x-3y=15. & (2) \end{cases}$$

Eliminate x.

From (1) we find $x=25-2y$; (3)

and from (2), $x=\frac{15+3y}{2}$. (4)

Equating the values of x in (3) and (4), $25-2y=\frac{15+3y}{2}$;

solving the equation, $y=5$.

Substituting 5 for y in (1), $x+10=25$;

whence $x=15$.

30. Solve the equations

$$\begin{cases} \frac{y}{4}+\frac{z}{3}=9, & (1) \\ \frac{y}{3}-\frac{z}{6}=1. & (2) \end{cases}$$

Eliminate y.

From (1) we find $y = 36 - \frac{4z}{3}$;

and from (2), $y = 3 + \frac{z}{2}$.

Equating the values of y, $3 + \frac{z}{2} = 36 - \frac{4z}{3}$;

solving the equation, $z = 18$.

Substituting 18 for z in (1), $\frac{y}{4} + 6 = 9$;

whence $y = 12$.

Solve the equations

31. $\begin{cases} 2x - y = 16, \\ x + 3y = 36. \end{cases}$

32. $\begin{cases} 3x + 4z = 18, \\ x + 2z = 8. \end{cases}$

33. $\begin{cases} 2y - 3z = -7, \\ 4y - 5z = -9. \end{cases}$

34. $\begin{cases} x + 4z = 11, \\ 7x - 3z = 15. \end{cases}$

35 $\begin{cases} y - z = 4, \\ 3y + 5z = 20. \end{cases}$

36. $\begin{cases} 4x + 12y = 5, \\ x + 6y = 2. \end{cases}$

37 $\begin{cases} y - 2z = -6, \\ 4y - 3z = 26. \end{cases}$

38. $\begin{cases} x + 3z = 50, \\ 3x - 2z = 7. \end{cases}$

39. $\begin{cases} \frac{x}{2}+\frac{2y}{3}=16, \\ \frac{x}{4}-\frac{y}{6}=2. \end{cases}$

40. $\begin{cases} x - z = 4, \\ \frac{3x}{7}+\frac{2z}{3}=5. \end{cases}$

41. $\begin{cases} \frac{x-4}{4}+\frac{y-3}{3}=3, \\ \frac{x-3}{3}-\frac{y-4}{4}=2\frac{1}{2}. \end{cases}$

42. $\begin{cases} ax - by = 10, \\ ax + by = 26. \end{cases}$

43. $\begin{cases} \frac{x}{4}+\frac{2y}{3}=12, \\ \frac{x}{2}-\frac{3y}{4}=-1. \end{cases}$

To eliminate by comparison,

Find from each of the given equations the value of one of the unknown numbers in terms of the other.

Equate the values of the unknown number thus found, and solve the resulting equation.

Miscellaneous Exercises.

301. The following pairs of simultaneous equations may be solved by any one of the above methods of elimination. The forms of the given equations will usually indicate which method will be the most advantageous.

302. It will sometimes be best to simplify the equations before determining the method of elimination to be used.

Thus, the equations

$$(1)\ \frac{x+y}{5}-\frac{x-y}{6}=2, \text{ and } (2)\ \frac{x+y}{2}+\frac{x-y}{3}=10,$$

may first be changed (1) to $x+11y=60$, and (2) to $5x+y=60$, and then either x or y may be eliminated by subtraction or by substitution.

The method by addition or subtraction will usually be found the most advantageous; but, when the coefficient of one of the unknown numbers in either equation is unity, the method by substitution may be preferable.

Solve the equations

1. $\begin{cases} x+y=19, \\ x-y=7. \end{cases}$

2. $\begin{cases} 2x+3y=13, \\ 5x+4y=22. \end{cases}$

3. $\begin{cases} 7x+4y=85, \\ 13x-37y=69. \end{cases}$

4. $\begin{cases} 11x+3y=100, \\ 4x-7y=4. \end{cases}$

5. $\begin{cases} 15x-13y=21, \\ 17x-y=65. \end{cases}$

6. $\begin{cases} 3x-5y=5, \\ 5x-3y=35. \end{cases}$

7. $\begin{cases} 21x-19y=-9, \\ 19x-21y=-31. \end{cases}$

8. $\begin{cases} 25x+21y=410, \\ 16x-14y=-12. \end{cases}$

9. $\begin{cases} 14x-13y=51, \\ 15y-x=123. \end{cases}$

10. $\begin{cases} x+2y=3, \\ 5y+4x=6. \end{cases}$

11. $\begin{cases} \frac{x}{2}+\frac{y}{3}=7, \\ \frac{x}{3}+\frac{y}{2}=8. \end{cases}$

12. $\begin{cases} x+\frac{1}{2}y=\frac{1}{3}, \\ \frac{1}{4}x+\frac{1}{5}y=\frac{1}{6}. \end{cases}$

13. $\begin{cases} 60x-17y=146, \\ 48x+13y=170. \end{cases}$

14. $\begin{cases} 4x+12y=5, \\ x+2y=1. \end{cases}$

15. $\begin{cases} \frac{3x}{7}+\frac{2y}{3}=5, \\ x-y=4. \end{cases}$

16. $\begin{cases} \frac{1}{5}x+3y=8, \\ \frac{1}{4}(3x-2)=2y+3. \end{cases}$

17. $\begin{cases} \frac{x}{7}+\frac{y}{14}=10\frac{1}{2}, \\ 2x-y=7. \end{cases}$

18. $\begin{cases} \frac{1}{3}(x+y)-\frac{1}{2}(x-y)=9, \\ \frac{1}{2}x+\frac{1}{3}(x+y)=5. \end{cases}$

19. $\begin{cases} \frac{1}{3}(x+y)-\frac{1}{6}(x-y)=2, \\ \frac{1}{2}(x+y)+\frac{1}{3}(x-y)=10. \end{cases}$

20. $\begin{cases} (x+5)(y+7)=(x+1)(y-9)+112, \\ 2x+10=3y+1. \end{cases}$

21. $\begin{cases} 5\frac{3}{4}y-11x=4y+117\frac{1}{8}, \\ 8x+175=2y. \end{cases}$

22. $\begin{cases} 49x-37\frac{4}{5}+48y=9x+273, \\ 3x-26\frac{1}{4}=-5y. \end{cases}$

23. $\begin{cases} 13x+7y=121, \\ 2x+\frac{1}{2}y=14. \end{cases}$

24. $\begin{cases} 10\frac{1}{2}x-5\frac{3}{4}y=40, \\ 9x-8.5y=20. \end{cases}$

25. $\begin{cases} .2x+.3y=4.3, \\ .2x-.2y=1.5. \end{cases}$

26. $\begin{cases} 1.7x-2.3y=1.6, \\ .4x+.06y=2.18. \end{cases}$

27. $\begin{cases} .2x+.4y=2.8, \\ .4x+.3y=3.1. \end{cases}$

28. $\begin{cases} \frac{x-2}{4}+\frac{y-2}{4}=1\frac{1}{2}, \\ \frac{x+2}{6}-\frac{y+2}{6}=1. \end{cases}$

29. $\begin{cases} \frac{x}{a}+\frac{y}{b}=1, \\ \frac{x}{b}+\frac{y}{a}=\frac{8}{4}. \end{cases}$

30. $\begin{cases} x+y=s, \\ x-y=d. \end{cases}$

31. $\begin{cases} ax - by = c, \\ x - y = d. \end{cases}$

32. $\begin{cases} mx - ny = z, \\ \dfrac{x}{m} - \dfrac{y}{n} = c. \end{cases}$

33. $\begin{cases} \dfrac{x}{m} + \dfrac{y}{n} = a, \\ \dfrac{x}{n} - \dfrac{y}{m} = b. \end{cases}$

34. $\begin{cases} \dfrac{x}{a+b} - \dfrac{y}{a-b} = 1, \\ \dfrac{x}{a+b} + \dfrac{y}{a-b} = 10. \end{cases}$

35. $\begin{cases} \dfrac{x}{m} - ny = m + n, \\ mx - \dfrac{y}{n} = m - n. \end{cases}$

36. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = c, \\ \dfrac{x}{b} = \dfrac{y}{a}. \end{cases}$

37. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2, \\ \dfrac{x}{c} + \dfrac{y}{c} = 1. \end{cases}$

38. $\begin{cases} \dfrac{x}{m} + \dfrac{y}{n} = 2, \\ \dfrac{x}{m} - \dfrac{y}{n} = 1. \end{cases}$

303. When simultaneous equations are fractional, and of the form $\frac{a}{x} \pm \frac{b}{y} = c$, they may be solved most readily by treating $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown numbers.

39. Solve the equations $\begin{cases} \dfrac{14}{x} + \dfrac{7}{y} = 4\frac{1}{2}, & (1) \\ \dfrac{21}{x} - \dfrac{6}{y} = 1. & (2) \end{cases}$

Eliminate x.

Multiplying (1) by 3, $\quad \dfrac{42}{x} + \dfrac{21}{y} = 13;$ (3)

multiplying (2) by 2, $\quad \dfrac{42}{x} - \dfrac{12}{y} = 2;$ (4)

subtracting (4) from (3), $\quad \dfrac{33}{y} = 11. \quad \therefore y = 3.$

Substituting 3 for y in (2), $\quad \dfrac{21}{x} = 3. \quad \therefore x = 7.$

Solve the equations

40. $\begin{cases} \frac{2}{x}+\frac{3}{y}=1, \\ \frac{12}{x}-\frac{12}{y}=1. \end{cases}$

41. $\begin{cases} \frac{1}{x}+\frac{3}{y}=\frac{3}{4}, \\ \frac{15}{x}-\frac{9}{y}=2\frac{1}{4}. \end{cases}$

42. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{8}{15}, \\ \frac{1}{x}-\frac{1}{y}=\frac{2}{15}. \end{cases}$

43. $\begin{cases} \frac{m}{x}+\frac{n}{y}=a, \\ \frac{n}{x}+\frac{m}{y}=b. \end{cases}$

44. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{5}{6}, \\ \frac{2}{x}-\frac{3}{y}=-\frac{5}{6}. \end{cases}$

45. $\begin{cases} \frac{7}{x}-\frac{5}{y}=\frac{1}{6}, \\ \frac{3}{x}+\frac{4}{y}=\frac{23}{21}. \end{cases}$

46. $\begin{cases} \frac{m}{x}+\frac{n}{y}=p, \\ \frac{n}{x}+\frac{m}{y}=q. \end{cases}$

47. $\begin{cases} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}-\frac{1}{y}=b. \end{cases}$

EQUATIONS WITH THREE OR MORE UNKNOWN NUMBERS.

304. When three equations with three unknown numbers are given, we may combine one of the equations with each of the other two in such manner as to obtain *two* equations with *two* unknown numbers. The two resulting equations may then be combined, as in the preceding cases, giving a single equation with one unknown number. The value of this unknown number being found, the value of the other unknown numbers may be obtained by substitution. This process may sometimes be shortened, as shown in Example 2.

1. Find the values of x, y, and z in

$$\begin{cases} 3x+5y+\ z=26, & (1) \\ 6x+3y+3z=36, & (2) \\ 9x+4y+4z=50. & (3) \end{cases}$$

First eliminate z.

Multiplying (1) by 3, $9x + 15y + 3z = 78$; (4)

subtracting (2) from (4), $3x + 12y = 42$. (5)

Multiplying (1) by 4, $12x + 20y + 4z = 104$; (6)

subtracting (3) from (6), $3x + 16y = 54$. (7)

Next eliminate x from (5) and (7).

Subtracting (5) from (7), $4y = 12$. $\therefore y = 3$.

Substituting 3 for y in (5), $3x = 6$. $\therefore x = 2$.

Substituting 3 for y, and 2 for x, in (1), $z = 5$.

Hence $x = 2$, $y = 3$, $z = 5$.

2. Solve the equations
$$\begin{cases} 2x + 3y + z = 25, & (1) \\ 4x + 5y + 2z = 46, & (2) \\ 6x + 4y + 4z = 58. & (3) \end{cases}$$

Multiplying (1) by 2, $4x + 6y + 2z = 50$; (4)

subtracting (2) from (4), $y = 4$.

Multiplying (1) by 3, $6x + 9y + 3z = 75$; (5)

subtracting (3) from (5), $5y - z = 17$; (6)

substituting 4 for y in (6), $z = 3$.

Substituting 4 for y, and 3 for z, in (1), $2x = 10$. $\therefore x = 5$.

305. When any one of the unknown numbers does not occur in all three equations, we may first eliminate such unknown number from the equations in which it does occur, thus obtaining two equations with two unknown numbers.

3. Solve the equations
$$\begin{cases} x + y = 11, & (1) \\ x + z = 10, & (2) \\ y + z = 9. & (3) \end{cases}$$

First eliminate x from (1) and (2)

Subtracting (2) from (1), $y - z = 1$. (4)

Adding (4) and (3), $2y = 10$. $\therefore y = 5$.

Substituting 5 for y in (3), $z = 4$.

Substituting 5 for y in (1), $x = 6$.

These equations may also be solved by first adding (1), (2), and (3), then dividing the resulting equation by 2, and from this equation subtracting successively (1), (2), and (3).

306. When four or more equations with four or more unknown numbers are given, one of the unknown numbers may be eliminated by combining one of the equations in which it occurs with each other equation in which it occurs; then a second unknown number may be eliminated by combining one of the resulting equations in which it occurs with each other resulting equation in which it occurs; and so on.

When each unknown number does not occur in all of the given equations, the process may be shortened. There must be as many given equations as there are unknown numbers.

4. Solve the equations
$$\begin{cases} x+y+z=9, & (1)\\ x+y-z+w=15, & (2)\\ x-y+z+w=11, & (3)\\ 3x-2y+4z=9. & (4) \end{cases}$$

First eliminate w.

(2) − (3), $2y-2z=4.$ (5)

Next eliminate x in (1) and (4).

(1) × 3, $3x+3y+3z=27;$ (6)

(6) − (4), $5y-z=18.$ (7)

(5) ÷ 2, $y-z=2;$ (8)

(7) − (8), $4y=16.$ $\therefore\ y=4.$

Substituting 4 for y in (8), $z=2.$

Substituting values of y and z in (1), $x=3.$

Substituting values of x, y, and z in (2), $w=10.$

Solve the equations

5. $\begin{cases} 3x+5y-z=18,\\ 5x-6y+5z=8,\\ x+y+z=6. \end{cases}$

6. $\begin{cases} 4x+2y-4z=22,\\ 5x-4y+2z=23,\\ 5z-x+3y=28. \end{cases}$

7. $\begin{cases} 2x+y+6z=46,\\ 6x+4y-3z=16,\\ 4x-6y+4z=12. \end{cases}$

8. $\begin{cases} 7x+3y-5z=3,\\ 3z-y-4x=1,\\ x+y-z=1. \end{cases}$

9. $\begin{cases} 3x-3y+4z=25,\\ 5x+2y-z=9,\\ 4y-11x+3z=11. \end{cases}$

10. $\begin{cases} 5x-4y+z=3,\\ 3x+6y-2z=14,\\ 4x+5y-z=18. \end{cases}$

11. $\begin{cases} 4x-7y+5z=7, \\ 6x+4y-6z=28, \\ 6y-z+7x=56. \end{cases}$

12. $\begin{cases} x-3y+2z=0, \\ 3x+2y-5z=22, \\ x-y-z=2. \end{cases}$

13. $\begin{cases} x+y-z=16, \\ x-y+z=6, \\ x-y-z=-2. \end{cases}$

14. $\begin{cases} \frac{1}{2}x-\frac{1}{3}y+\frac{1}{4}z=4, \\ \frac{1}{3}x+\frac{1}{4}y-\frac{1}{5}z=1, \\ \frac{1}{4}z-\frac{1}{6}y-\frac{1}{2}x=0. \end{cases}$

15. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62, \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47, \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38. \end{cases}$

16. $\begin{cases} \dfrac{5x-3y}{4z-7x}=1, \\ \dfrac{3z-x}{3y+x}=1, \\ \dfrac{3x+5y}{4z+5}=1. \end{cases}$

17. $\begin{cases} \dfrac{x+y-z}{x-y-z}=8, \\ \dfrac{y+x+z}{y-x-z}=7, \\ y-x+z=-2. \end{cases}$

18. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1, \\ \dfrac{x}{a}+\dfrac{y}{c}+\dfrac{z}{b}=1, \\ \dfrac{x}{b}+\dfrac{y}{a}+\dfrac{z}{c}=1. \end{cases}$

19. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x}+\dfrac{1}{z}=b, \\ \dfrac{1}{y}+\dfrac{1}{z}=c. \end{cases}$

20. $\begin{cases} 7x-3y=1, \\ 11z-7v=1, \\ 4z-7y=1, \\ 19x-3v=1. \end{cases}$

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS.

NOTE. In the statement of a problem there must be as many equations as there are *letters representing unknown numbers* (§ 306).

1. The sum of two numbers is 343, and their difference is 49. What are the numbers?

2. Divide the number 89 into two parts such that $\frac{1}{3}$ of the greater part will exceed $\frac{1}{2}$ of the less by unity.

3. A boy, being asked his age and that of his sister, replied, "If I were 3 years older, I would be 3 times as old as my sister; but, if she were 2 years older, she would be $\frac{1}{2}$ as old as I am." How old was each?

4. If A were a years older than he is, he would be m times as old as B; but, if B were b years older than he is, he would be $\frac{1}{n}$ the age of A. How old is each?

5. A man has $10,000 in two investments. From the first he gets 6% interest; and from the second, 4%; yet the second investment yields twice as much income as the first. What is the amount of each investment?

6. In an alloy of silver and copper, $\frac{5}{6}$ of the whole, and 42 ounces more, was silver; while the copper was 8 ounces less than $\frac{1}{9}$ of the whole. How many ounces were there of each?

7. In an alloy of nickel, copper, and silver, $\frac{1}{5}$ of the whole, plus 8 ounces, is nickel; $\frac{2}{5}$ of the whole, plus 4 ounces, is copper; and $\frac{1}{7}$ of the whole is silver. How many ounces of each metal in the alloy?

8. A certain number expressed by two digits is equal to 4 times the sum of those digits; but, if 18 be added to the number, the digits will be reversed. Find the number.

9. Said A to B, "If you give me $100, I shall then have as much money as you."—"Nay," replied B, "give me $100, and then I shall have 3 times as much as you." How much money had each?

10. If 10 lb. of tea, with 35 lb. of sugar, cost $11.30, and 12 lb. of tea with 25 lb. of sugar cost $12.20, what is the price of each per pound?

11. If a lb. of tea with b lb. of sugar cost m cents, but c lb. of tea with d lb. of sugar cost n cents, what is the cost of each per pound?

12. If the numerator of a certain fraction be multiplied by 2, and its denominator increased by 2, the result will be equal to unity; but if the denominator be multiplied by 2, and the numerator increased by 3, the result will be equal to $\frac{1}{2}$. What is the fraction?

13. A certain fraction becomes equal to $\frac{4}{5}$ when 3 is added to both its terms, but becomes equal to $\frac{5}{7}$ when the same number is taken from both its terms. Find the fraction.

14. A certain fraction becomes equal to $\frac{3}{4}$ if 2 be taken from its numerator, and it becomes equal to unity if 3 be taken from its denominator. Find the fraction.

15. A certain fraction which is equal to $\frac{4}{5}$ is increased to $\frac{5}{6}$ by having the same number added to both its terms, and is multiplied by 2 by having another number taken from both its terms. Find the numbers.

16. A fishing rod consists of two parts. The length of the upper part is to that of the lower part as 5 to 7; and 9 times the upper part, with 13 times the lower part, is 36 inches more than 11 times the length of the whole rod. Find the length of each part.

17. A dealer has two sorts of tea. By mixing them at the rate of 3 lb. of the finer to 5 lb. of the coarser, he can sell the mixture at 95 cents a pound; but, mixing at the rate of 1 lb. of the finer to 3 lb. of the coarser, he can sell the mixture at 90 cents a pound. What is the price per pound of each kind of tea?

18. The difference of two numbers is 8, and twice the sum of their reciprocals is equal to 3 times the difference of their reciprocals. Find the numbers.

19. Seven years ago A's age was just 3 times that of B, but 7 years hence A's age will be just double that of B. What is the age of each?

20. A person, wishing to give 25 cents each to a certain number of persons, found he had not enough money by 25 cents; but he could give each 23 cents and have 1 cent left. How many persons were there, and what sum had he?

21. A cistern can be filled in 6 hours by two pipes running together, and the one pipe fills as much of the cistern in $\frac{3}{4}$ of an hour as the other does in 1 hour. In what time could they separately fill the cistern?

22. There are three numbers, such that the first together with $\frac{2}{3}$ of the second is equal to 19; $\frac{1}{3}$ of the second with $\frac{3}{4}$ of the third is equal to 23; and $\frac{1}{2}$ of the third with $\frac{1}{3}$ of the first is equal to the second. What are these numbers?

23. A certain number has three digits whose sum is 15. The digit in the units place is 3 times that in the hundreds place; and, if 396 be added to the number, the order of these digits will be reversed in the result. What is the number?

24. A and B together earn \$ 50 in 8 days; A and C together, \$ 69 in 12 days; B and C together, \$ 55 in 10 days. How much can each earn in a day?

25. A jeweler sold three rings. The price of the first with $\frac{1}{2}$ that of the second and third was \$ 25; the price of the second with $\frac{1}{3}$ that of the first and third was \$ 26; and the price of the third with $\frac{1}{2}$ that of the first and second was \$ 29. What was the price of each?

26. A person finds he can buy with \$ 31.10 either 10 bushels of wheat, 12 of rye, and 9 of oats; or 12 bushels of wheat, 6 of rye, and 13 of oats; or 16 bushels of wheat, 10 of rye, and 2 of oats. What is the price of each grain per bushel?

27. There are four numbers such that, by adding each to twice the sum of the remaining three, we obtain 46, 43, 41, and 38 respectively. What are the numbers?

28. A man has two horses, and also a saddle worth \$ 10. If he puts the saddle on the first horse, his value will be double that of the second horse; but, if he puts the saddle on the second horse, his value will be \$ 13 less than that of the first. What is the value of each horse?

CHAPTER XII.

INVOLUTION AND EVOLUTION.

POWERS.

307. The **power** of a number is the product obtained by taking the number one or more times as a factor (§ 27). The factor taken one or more times is called the **base** of the power.

308. The exponent of a number denotes the **degree** of its power (§ 29). The exponent 2 denotes the second degree; the exponent 3, the third degree; and the exponent n, the nth degree.

309. A number that is composed entirely of equal factors is a **perfect power**; and a number that is not composed of equal factors is an **imperfect power.**

310. **Involution** is the process of raising numbers to required powers.

Since any power of a number is the continued product of the number by itself, involution involves the principles of multiplication.

311. $25^2 = 25 \times 25$; $(3a)^3 = 3a \times 3a \times 3a$; and $(3a)^n = 3a \times 3a \times 3a \cdots$ to n factors. Hence

Any power of a number may be found *by taking the number as many times as a factor as there are ones in the exponent of the required power.*

1. What is the fourth power of 4? Of 5? Of 10?

2. What is the third power of a? Of x? Of y?

312. $(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^{2\times 3} = a^6$; and $(a^m)^n = a^m \times a^m \times a^m \cdots$ to n factors $= a^{m\times n} = a^{mn}$. Hence

Any power of a number may be raised to any required power *by multiplying the exponent of the given power by the exponent of the required power.*

3. What is the cube of a^2? Of x^3? Of y^4?

4. What is the nth power of a^2? Of x^3? Of y^4?

5. What is the nth power of a^m? Of b^n? Of x^r?

Monomials.

313. $12^3 = (3 \times 4)^3 = 3^3 \times 4^3$; $(3a)^3 = 3^3 \times a^3$; and $(3a)^n = 3^n \times a^n$. Hence

Any power of a number is *the product of each of the factors of the number raised to the required power.*

1. What is the third power of 2×5? Of 4×3?

2. What is the fourth power of ax^2? Of a^2b^3x?

3. What is the fifth power of $2ab^2c$? Of $3a^2xy^3$?

314. It is thus seen that *any power of a monomial may be found by raising each of its factors to the required power.*

315. $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a \times a}{b \times b} = \frac{a^2}{b^2}$; $\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$; and $\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \cdots$ to n factors $= \frac{a^n}{b^n}$. Hence

A fraction is raised to any power *by raising both its terms to the required power.*

4. What is the fourth power of $\frac{a^2}{b}$? Of $\frac{a}{x^2}$?

5. What is the nth power of $\frac{x}{7}$? Of $\frac{x^n}{y^m}$?

316. $(+a)^3=(+a)\times(+a)\times(+a)=+a^3$; and $(+a)^n=+a^n$. $(-a)^2=(-a)\times(-a)=+a^2$; $(-a)^3=(+a^2)\times(-a)=-a^3$; $(-a^3)\times(-a)=+a^4$; $(+a^4)\times(-a)=-a^5$; and so on. Hence

I. If a number is positive, all its powers are positive.

II. If a number is negative, its *even* powers are positive, and its *odd* powers negative.

317. This principle may be expressed by the formulas $(+a)^n=+a^n$; $(-a)^{2n}=+a^{2n}$; and $(-a)^{2n+1}=-a^{2n+1}$

318. It is thus seen that *all even powers of a number are positive, and all odd powers have the sign of the number itself.*

It follows that $-a^2$, $-a^4$, etc., are not perfect powers. See "Imaginary Numbers" (§ 413). The term *number* in §§ 316, 318, denotes a *real* number; i.e., a number that is not imaginary.

6. What is the third power of $-3a^2b$? Of $4ab^2$?

7. What is the third power of $\frac{3a}{2b}$? Of $-\frac{2a}{3x}$? Of $-\frac{4}{3ax}$?

Raise to the indicated power

8. $(3a^2b)^3$.
9. $(5a^2xy^3)^2$.
10. $(-4ab^2x)^3$.
11. $(-a^2bcx^2)^5$.
12. $(3a^3b^2c)^6$.
13. $(-2a^3xy^2)^3$.
14. $(-3ab^{\frac{1}{2}}x^{\frac{3}{4}}y^3)^4$.
15. $(a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{3}})^{12}$.
16. $(a^nb^mc)^2$.
17. $(-2a^mx^ny^r)^n$.
18. $(3abx)^{2n}$.
19. $(-xyz)^{2n}$.
20. $(-abc)^{2n+1}$.
21. $(-3x^my^{2m})^3$.
22. $\left(\frac{3a^2x}{-4y}\right)^2$.
23. $\left(\frac{-2x^2y^3}{5ab^2}\right)^3$.
24. $\left(\frac{-5xy}{-3z^2}\right)^4$.
25. $\left(\frac{abc}{x}\right)^n$.
26. $\left(\frac{a^2b^2}{xy}\right)^m$.
27. $\left(-\frac{a}{xy}\right)^{2n}$.
28. $\left(-\frac{x}{a}\right)^{2n+1}$.

319. To raise a monomial to any power,

Raise the numerical coefficient to the required power, multiply the exponent of each literal factor by the exponent which indicates the required power, and prefix the proper sign.

320. A fraction is raised to any power *by raising both terms to the required power, and prefixing the proper sign.*

Binomials.

321. It may be found by actual multiplication that

$$(a+b)^2 \equiv a^2 + 2ab + b^2.$$
$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a+b)^4 \equiv a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$
$$(a+b)^5 \equiv a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$
$$(a-b)^2 \equiv a^2 - 2ab + b^2.$$
$$(a-b)^3 \equiv a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a-b)^4 \equiv a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$
$$(a-b)^5 \equiv a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

322. An inspection of the above powers of $a+b$ and $a-b$ shows the following facts or laws:

I. The number of *terms* in each power of the binomial is one more than the exponent of the power.

II. The *exponent* of a in the first term is the same as the exponent of the power of the binomial, and it decreases by 1 in each succeeding term.

III. The *exponent* of b in the second term is 1, and it increases by 1 in each succeeding term.

b does not appear in the first term, nor a in the last term.

The *sum* of the exponents of a and b in each term is the exponent of the power of the binomial.

IV. The *coefficient* of the first term is 1, and the coefficient of the second term is the same as the exponent of the power of the binomial; and generally the coefficient of any term is the product of the coefficient of the preceding term by the exponent of a in that term, divided by the number of that term.

For example, in the fourth power of $a + b$, the coefficient of the third term is $4 \times 3 \div 2 = 6$.

The coefficients of the first, second, third, etc., terms from the left are respectively equal to the coefficients of the first, second, third, etc., terms from the right.

V. When both a and b are positive, all the terms of the power are positive; and when b is negative, the *odd* terms are positive, and the *even* terms are negative.

323. It is to be noted that the above equations denoting the powers of $a + b$ and $a - b$ are *identities*, and are true for all values of a and b.

324. It will be shown later (Chapter XXI.) that the foregoing laws hold good for any power of a binomial indicated by a positive integral exponent. Hence, if n denotes any positive integer,

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots.$$

This identity is called the **binomial formula.**

325. A binomial may be expanded to any power indicated by a positive integral exponent by the application of the above formula.

1. Expand $(x-y)^5$.

What is the number of terms in the expansion?

What is the exponent of x in the first term? In the third term? In the last term but one?

What is the exponent of y in the second term? In each sequent term?

What is the coefficient of the first term? Of the second term? Of each sequent term?

How is the coefficient of the third term found? Of the fourth term?

What is the sign of each term?

Expand in like manner

2. $(a-b)^6$. 5. $(a-x)^5$. 8. $(a+b)^6$.
3. $(m-n)^4$. 6. $(x+y)^7$. 9. $(1-x)^3$.
4. $(a+x)^5$. 7. $(c-d)^4$. 10. $(x-1)^4$.

Write the fourth term of

11. $(a-b)^8$. 12. $(c+d)^{10}$. 13. $(x-y)^9$.

Write the first three terms of

14. $(a-b)^8$. 15. $(c+d)^9$. 16. $(x-y)^{10}$.

Write the last three terms of

17. $(m-n)^6$. 18. $(a+x)^8$. 19. $(a-b)^7$.

326. When one or both terms of a binomial contain more than one factor, as $3a-b^2$, the binomial may in like manner be raised to any power.

20. What is the third power of $2x^2-3y$?

Let $a=2x^2$, and $b=3y$.

Formula. $(a-b)^3 \equiv a^3-3a^2b+3ab^2-b^3$.

Substituting $2x^2$ for a, and $3y$ for b,

$$(2x^2-3y)^3=(2x^2)^3-3(2x^2)^2(3y)+3(2x^2)(3y)^2-(3y)^3$$
$$=8x^6-36x^4y+54x^2y^2-27y^3.$$

NOTE. This is in effect the same as the inclosing of each term containing more than one factor in a parenthesis, and treating it as a or b.

327. Since each term, the first and last excepted, of a binomial power higher than the first power, is composed of *three* factors, the expansion of such a binomial power as $(2x^2-3y)^3$ may be somewhat facilitated by writing these three factors in a column, and then forming their product as below.

Formula.

$$\begin{array}{llll} (a-b)^3 \equiv a^3 & -3a^2b & +3ab^2 & -b^3. \\ (2x^2)^3 & -3 & +3 & -(3y)^3 \\ & (2x^2)^2 & 2x^2 & \\ & 3y & (3y)^2 & \end{array}$$

$$(2x^2-3y)^3=8x^6-36x^4y+54x^2y^2-27y^3$$

Expand as above

21. $(3a - 2b^2)^3$. 23. $\left(ab - \frac{c}{2}\right)^4$. 25. $(2m - n^5)^5$.

22. $(a^2 - 3bc)^3$. 24. $\left(\frac{a}{2} - bc\right)^4$. 26. $\left(\frac{ab}{2} - 4c\right)^3$.

Polynomials.

328. Polynomials of three or four terms may in like manner be raised to any power.

1. What is the third power of $2a^2 + b - c$?

$$2a^2 + b - c = 2a^2 + (b - c).$$

Let $x = 2a^2$, and $y = b - c$.

Formula. $(x + y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$

Substituting, $(2a^2)^3 + 3 (2a^2)^2 (b - c) + 3 (2a^2) (b - c)^2 + (b - c)^3$

$$(2a^2 + b - c)^3 = 8a^6 + 12a^4b - 12a^4c + 6a^2b^2 - 12a^2bc + 6a^2c^2 + b^3 - 3b^2c + 3bc^2 - c^3.$$

2. Expand $(a + b + c - d)^3$.

$$a + b + c - d = (a + b) + (c - d).$$

Let $x = a + b$, and $y = c - d$.

Formula. $(x + y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$.

Substituting, $(a + b)^3 + 3 (a + b)^2 (c - d) + 3 (a + b) (c - d)^2 + (c - d)^3$

$$(a + b + c - d)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c - 3a^2d - 6abd - 3b^2d + 3ac^2 - 6acd + 3ad^2 + 3bc^2 - 6bcd + 3bd^2 + c^3 - 3c^2d + 3cd^2 - d^3.$$

329. Since the square of a polynomial is composed of (1) the square of each term, and (2) twice the product of each term multiplied in succession by the terms which follow it (§ 161), the square of a polynomial may be directly written.

Thus, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2a(b + c) + 2b \times c$

$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

The *cube* of a trinomial is composed of (1) the cube of each term; (2) three times the product of the square of each term multiplied by the sum of the other two terms; and (3) six times the product of the three terms.

Raise to the indicated power

3. $(5\,a^2xy^3)^2$.
4. $(6\,ab^2c^3x^4)^3$.
5. $(-2\,a^2x^3y^4)^5$.
6. $(-3\,m^3nx^2y)^4$.
7. $(-p^2xy^3)^7$.
8. $(-dx^2yz^5)^6$.
9. $\left(\frac{a^2}{m^3}\right)^2$.
10. $\left(-\frac{c^2x^3}{3\,ny}\right)^4$.
11. $\left(\frac{-2\,a^2c^3}{2\,mn^2}\right)^3$.
12. $\left(-\frac{x^2y^2z}{2\,ab^3c}\right)^5$.
13. $\left(\frac{-ab^3c}{xy^2z}\right)^n$.
14. $\left(\frac{x^my^n}{a^mc^n}\right)^m$.

Expand by the binomial formula

15. $(m+n)^4$.
16. $(m-n)^5$.
17. $(a^2-b)^3$.
18. $(x-1)^4$.
19. $(1-x)^3$.
20. $(1-z^2)^4$.
21. $(x^3-3\,a^2)^3$.
22. $(y^4-3\,z^2)^3$.
23. $\left(2\,a+\frac{x}{2}\right)^4$.
24. $\left(\frac{ab}{3}-c^2\right)^3$.
25. $\left(x-\frac{a}{x}\right)^4$.
26. $\left(x^2+\frac{x}{a^2}\right)^3$.

Write the first three and the last three terms of

27. $(x+y)^{15}$.
28. $(x-a^2)^{10}$.
29. $(x+y)^{12}$.

Write the sixth sequent term of

30. $(a-c)^{12}$.
31. $(1-x)^{10}$.
32. $(x-1)^9$.

ROOTS.

330. The **root** of a number is one of the equal factors which multiplied together will produce the number (§ 34).

331. The *second* or square root of a number is one of its *two* equal factors; the *third* or cube root, one of its three equal factors; and the nth root, one of its n equal factors. Hence the nth root of a^n is a; i.e., the root of any power is its base.

332. The root of a number may be indicated by the radical sign, $\sqrt{\ }$. Thus, $\sqrt[2]{25}$ or $\sqrt{25}$ denotes the square root of 25; $\sqrt[3]{64}$, the cube root of 64; and $\sqrt[n]{ab}$, the nth root of ab (§ 35).

The figure or letter written in the opening of the radical sign is called the *index* of the root.

The root of a number may also be indicated by a *fractional exponent*. Thus, $a^{\frac{1}{2}}$ denotes the square root of a; $a^{\frac{1}{n}}$, the nth root of a; and $a^{\frac{m}{n}}$, the nth root of the mth power of a (§ 371).

333. Evolution is the process of finding the required root of a number.

Evolution is the inverse of involution. Involution finds the product of equal factors, and evolution finds one of the equal factors of a product.

Monomials.

334. Since $(3\,a^2b^3)^3 = 3^3\,a^{2\times3}b^{3\times3}$ (§ 319) $= 27\,a^6b^9$,

$$\sqrt[3]{27\,a^6b^9} = \sqrt[3]{27} \times a^{6\div3}b^{9\div3} = 3\,a^2b^3.$$

Hence any root of a positive monomial is found *by taking the required root of its numerical coefficient, and then dividing the exponent of each literal factor by the index of the required root.*

Write the indicated root of

1. $\sqrt[3]{125\,a^3x^6}$. 2. $\sqrt[5]{32\,x^{10}y^5z^{15}}$. 3. $\sqrt[3]{64\,m^6n^3x^9}$.

335. Since any *even* power of $\pm a$ is positive (§ 316), any *even* root of a positive number is $+$ or $-$. Thus,

$$\sqrt[4]{16\,x^4} = \pm 2\,x.$$

Since $(+a)^2 = +a^2$, and $(-a)^2 = +a^2$, the square root of $-a^2$ is neither $+a$ nor $-a$. The square root of $-a^2$ can only be indicated ($\sqrt{-a^2}$), and the same is true of any *even* root of a *negative* number. Such an indicated root of a negative number is called *unreal* or *imaginary* (§ 413).

A number has as many roots as there are ones in the *index* of the root. Thus, $\sqrt[4]{16\,x^8} = \pm 2\,x^2$ and $\pm\sqrt{-4\,x^4}$. The first two roots are *real*, and the second two roots ($\pm\sqrt{-4\,x^4} = \pm 2\,x^2\sqrt{-1}$) are *unreal*. Only real roots are given in this chapter.

336. Since any odd power of $+a$ is positive, and any odd power of $-a$ negative, any odd root of a positive number is positive, and any odd root of a negative number is negative.

337. It is thus seen that *the sign of the* EVEN *root of a positive number is* $+$ *or* $-$, *and that the* ODD *root of any number has the same sign as the number itself.*

338. Since $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $\sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}$. Hence the root of an algebraic fraction is found *by taking the required root of both its terms, and prefixing the proper sign.*

Write the indicated roots of

4. $\sqrt{25\,a^2b^2}$.
5. $\sqrt{81\,a^4b^6c^8}$.
6. $\sqrt[4]{256\,x^8y^4}$.
7. $\sqrt[3]{64\,a^3b^6c^9}$.
8. $\sqrt[3]{-27\,m^3n^6}$.
9. $\sqrt[6]{64\,a^6b^{12}x^{24}}$.
10. $\sqrt[3]{-125\,m^3n^6s^{12}}$.
11. $\sqrt[4]{81\,x^8y^{12}z^4}$.
12. $\sqrt[5]{-243\,a^{10}c^{20}}$.
13. $\sqrt[5]{-32\,a^{10}x^5}$.
14. $\sqrt[m]{a^mb^m}$.
15. $\sqrt[n]{a^{mn}x^{2n}}$.
16. $\sqrt[2n]{x^{2n^2}y^{2mn}}$.
17. $\sqrt[n]{x^{4n}y^{3n}}$.
18. $\sqrt{\frac{16\,a^4b^2}{25\,x^2y^4}}$.
19. $\sqrt[3]{-\frac{216\,a^3x^6}{125\,b^6c^{12}}}$.
20. $\sqrt[5]{-\frac{32\,a^5b^{10}}{x^{10}y^5}}$.
21. $\sqrt[4]{\frac{81\,a^8x^4}{256\,m^{12}n^{16}}}$.
22. $\sqrt[n]{\frac{a^{mn}x^{3n}}{y^{2n}z^n}}$.
23. $\sqrt[m]{\frac{x^{m^2}y^{2m}}{z^{m^2}}}$.
24. $\sqrt[3]{-\frac{x^{3m}y^{6n}}{a^6b^{3m}}}$.

SQUARE ROOTS OF POLYNOMIALS.

339. Since the square of $a \pm b$ is $a^2 \pm 2ab + b^2$, it follows that any trinomial is a perfect square, if, when properly arranged, its extreme terms are perfect squares, and its middle term twice the product of the square roots of its extreme terms (§ 186). Hence

The square root of a trinomial that is a perfect square may be readily found *by arranging the terms according to some letter, and then connecting the square roots of the extreme terms with the sign of the middle term.*

Find by inspection the square root of

1. $x^2 - 4xy + 4y^2$.
2. $4x^4 + 4x^2y + y^2$.
3. $x^4 - 6x^2y^2 + 9y^4$.
4. $9a^2 - 12ay^2 + 4y^4$.
5. $4a^2 - 4a + 1$.
6. $1 + 6y^3 + 9y^6$.
7. $9x^2 - 2xy + \frac{1}{9}y^2$.
8. $\frac{1}{4}x^2 - xy^2 + y^4$.

340. The second term of the square root of a trinomial which is a perfect square may also be found by dividing the middle term of the trinomial, properly arranged, *by twice the square root of its first term.*

341. Since the square of any polynomial is composed of (1) the square of each of its terms, and (2) twice the product of each term multiplied in succession by the terms which follow it (§ 329), the square root of a polynomial that is a perfect square may be found by inspection, provided the given polynomial contains *only two different powers of some particular letter.* Thus, the square root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$ is $a + b + c$; and this may be verified by squaring $a + b + c$.

342. It may assist, in finding the required root, if the polynomial be first arranged according to the descending powers of some particular letter. Thus, $a^2 + b^2 + c^2 + 2bc + 2ac + 2ab$, when arranged according to the descending powers of a, becomes

$$a^2 + 2a(b + c) + b^2 + 2b(c) + c^2.$$

Find by inspection the square root of

9. $x^2 + y^2 + 4 + 2xy - 4x - 4y$.
10. $x^4 + 4x^2y^2 + 6x^2z + 8x^2 + 4y^4 + 12y^2z + 16y^2 + 9z^2 + 24z + 16$.

343. Since $(a^2 + 2ab + 3b^2)^2 = (a^2)^2 + 2a^2(2ab + 3b^2) + 4a^2b^2 + 2(2ab \times 3b^2) + 9b^4 = a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4$, the square root of $a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4$ is found *by reversing the foregoing process,* as is shown below.

11. What is the square root of

$$a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4?$$

$a^4+4a^3b+10a^2b^2+12ab^3+9b^4 \mid a^2+2ab+3b^2$, sq. root.

a^4

$2a^2+ \mid +4a^3b+10a^2b^2$ (1)

$(2a^2+2ab)2ab = +4a^3b+4a^2b^2$

$2a^2+ \mid 6a^2b^2+12ab^3+9b^4$ (2)

$(2a^2+4ab+3b^2)3b^2 = 6a^2b^2+12ab^3+9b^4$

It is seen that the first term of the root (a^2) is the square root of the first term of the polynomial; that the second term of the root ($+2ab$) is obtained by dividing the first term of the first remainder (1) by twice the first term of the root, and the third term ($3b^2$) by dividing the first term of the second remainder (2) by twice the first term of the root.

NOTE. If preferred, twice the first term of the root ($2a^2$) may be considered the first "trial divisor;" and twice the first two terms of the root ($2a^2+4ab$), the second "trial divisor."

The first subtrahend (a^4) is the square of the first term of the root; the second subtrahend is twice the first term, plus the second term, multiplied by the second term $[(2a^2+2ab)2ab]$; and the third subtrahend is twice the first two terms of the root, plus the third term, multiplied by the third term $[(2a^2+4ab+3b^2)3b^2]$.

For verification, square the root found by the binomial formula, or, if preferred, by § 329, rearranging the terms of the result, if necessary, and reducing to the simplest form.

Find the square root of

12. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

13. $9x^4 - 12x^3y + 34x^2y^2 - 20xy^3 + 25y^4$.

14. $50a + 15a^2 + 25 + a^4 - 10a^3$.

15. $1 - 10x + 27x^2 - 10x^3 + x^4$.

16. $4x^2 + x^2y^2 + \frac{1}{4}y^2 - 4x^2y + 2xy - xy^2$.

17. $1 - 4x + 6y + 4x^2 - 12xy + 9y^2$.

18. $81 - 18a + a^2 + 18b - 2ab + b^2$.

19. $\frac{1}{4}x^4 + \frac{2}{3}x^2y + \frac{3}{4}x^2 + \frac{4}{9}y^2 + y + \frac{9}{16}$.

20. $\frac{4x^2}{9} - 2xy + \frac{9y^2}{4}$. 21. $x^2 + y^2 + \frac{4}{9} - 2xy + \frac{4x}{3} - \frac{4y}{3}$.

22. $1 + 2x + 2y + 2z + x^2 + 2xy + 2xz + y^2 + 2yz + z^2$.

23. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 4x^5 + 3x^6 - 2x^7 + x^8$.

24. $a^6 - 6a^4 + 15a^2 - 20 + \frac{15}{a^2} - \frac{6}{a^4} + \frac{1}{a^6}$.

344. To extract the square root of a polynomial,

Arrange the terms according to the powers of some letter.

Find the square root of the first term, and write it as the first term of the required root, and subtract its square from the first term of the polynomial.

Divide the first of the remaining terms by twice the first term of the root, and write the quotient as the second term of the root.

Multiply twice the first term of the root plus the second term by the second term, and subtract the product from the remaining terms; and so on until no term remains.

Square Roots of Numbers expressed by Figures.

345. The smallest integer expressed by one figure is 1, and the greatest is 9; the smallest integer expressed by two figures is 10, and the greatest is 99; and so on.

346. The squares of the smallest and the greatest integers expressed by one, two, and three figures, are as follows:

$1^2 = 1$	$10^2 = 100$	$100^2 = 10000$
$9^2 = 81$	$99^2 = 9801$	$999^2 = 998001$

It is thus seen that the square of an integer contains *twice as many orders as the integer, or twice as many orders less one.*

347. The squares of the smallest and the greatest numbers composed wholly of units, or tens, or hundreds, are as follows:

$1^2 = 1$	$10^2 = 100$	$100^2 = 10000$
$9^2 = 81$	$90^2 = 8100$	$900^2 = 810000$

It is thus seen that the square of *units* is units, or units and tens; the square of *tens* is hundreds, or hundreds and thousands; the square of *hundreds* is ten-thousands, or ten-thousands and hundred-thousands; and so on.

348. It follows, that if an integer be separated into periods of two orders each, beginning at the right, there will be as many orders in the root as there are periods in the integer; and hence *the square root of the left-hand period of the integer is the left-hand order of its square root.*

1. How many orders in the square root of 64? Of 625? Of 1444? Of 273529? Of 45796?

2. What orders in 273526 contain the square of the units of its square root? The square of the tens? The square of the hundreds? What is the first or left-hand figure of the root?

3. How many orders in the square root of 145796? What is the first figure of the root?

349. An integer may be separated into periods of two figures each by beginning at units, and placing a dot over each alternate figure, thus, $1\dot{4}5\dot{7}9\dot{6}$; or by placing the dot between the periods, thus, 14·57·96.

Separate into periods and give the first figure of the square root of

4. 625.	6. 94249.	8. 5306845.
5. 3025.	7. 492804.	9. 54756090.

350. The square of an integer may be found by the binomial formula, as shown below.

10. What is the square of 85?

$85 = 80 + 5.$

Let $a = 80$, and $b = 5$.

Since $(a+b)^2 \equiv a^2 + 2ab + b^2$,

$(80+5)^2 = 80^2 + 2 \times 80 \times 5 + 5^2$

$80^2 = 6400$

$2 \times 80 \times 5 = 800$

$5^2 = 25$

$= 7225$

It is thus seen that the square of any number composed of tens and units equals (1) *the square of the tens*, (2) *plus twice the product of the tens multiplied by the units*, (3) *plus the square of the units.*

351. Since the square of tens gives no order lower than *hundreds*, and the product of the tens by units gives no order lower than *tens*, the tens and units of the root may be found as shown below.

11. What is the square root of 7225?

$7\dot{2}2\dot{5} \mid \overset{t\ u}{8\,5}$

$a^2 = \quad 8^2 = \quad 64$

$2a = \quad 2 \times 80 = 160) 825$

$(2a+b)\,b = (160+5) \times 5 = \quad 825$

The same result may be obtained by omitting the unit figure in both the trial divisor (160) and the remainder (825): thus,

$7\dot{2}2\dot{5} \mid \overset{t\ u}{8\,5}$

$8^2 = \quad 64$

Trial divisor, $16) 82 \mid 5$

$165 \times 5 = \quad 82\,5$

12. What is the square root of 104976?

$1\dot{0}\,4\dot{9}\,7\dot{6} \mid \overset{h\ t\ u}{3\,2\,4}$

$(a = \overset{h}{3})\ 3^2 = \quad 9$

Trial divisor, $2 \times 3 = 6) 14 \mid 9$

$(20 \times 3 + 2)\,2 = 62 \times 2 = \quad 12\,4$

$(a = \overset{t}{32})$ Trial divisor, $32 \times 2 = 64) 2\,57 \mid 6$

$(320 \times 2 + 4)\,4 = 644 \times 4 = \quad 2\,57\,6$

It must be observed that a represents first 3 hundreds, considered as *3 tens with respect to the next figure of the root;* and that, in finding the third figure of the root, a represents 32 tens, *the part of the root already found.*

352. Since $.4^2 = .16$, $.04^2 = .0016$, etc., the square of a decimal has twice as many decimal orders as the decimal; and hence the square root of a decimal has one half as many decimal orders as the decimal. A decimal is separated into periods by beginning at the decimal point, and pointing off to the right; thus, .0625. A decimal cipher must be added if the decimal contains an *odd* number of orders; thus, .6250.

13. What is the square root of 13.3225?

$$\begin{array}{rl} & 13.3225 \mid 3.65 \\ 3^2 = & \quad 9 \\ 3 \times 2 = & 6)4.32 \\ 6.6 \times 6 = & \quad 3.96 \\ 3.6 \times 2 = & 7.2).3625 \\ 7.25 \times .05 = & \quad .3625 \end{array}$$

When an integer or a decimal is not a perfect square, its root may be found approximately by adding periods of decimal ciphers. Thus,

$$\sqrt{32} = \sqrt{32.0000} = 5.65+.$$

353. The square root of a common fraction is found by extracting the square root of each of its terms. Thus, $\sqrt{\frac{4}{9}} = \frac{2}{3}$.

When the denominator of a common fraction is not a perfect square, its square root can be found approximately by multiplying both of its terms by the denominator, and then extracting the square root of both terms of the resulting fraction, carrying the root of the numerator to two or more decimal places. Thus, $\sqrt{\frac{4}{7}} = \sqrt{\frac{4 \times 7}{7^2}} = \frac{\sqrt{28}}{7} = \frac{5.29}{7} = .75+.$

A common fraction may also be changed to a decimal, and the square root of the decimal found.

354. When the right-hand period of a decimal contains only one order, a decimal cipher should be annexed, 4.322 thus becoming 4.3220.

Find the square root of

14. 69169.	21. 176.89.	28. $\frac{256}{824}$.
15. 94249.	22. 45.1584.	29. $272\frac{1}{4}$.
16. 57600.	23. .008836.	30. $1040\frac{1}{16}$.
17. 210681.	24. .000625.	31. $\frac{7}{20}$.
18. 492804.	25. 75.364.	32. $\frac{8}{15}$.
19. 522729.	26. 586.7.	33. $\frac{2025}{2809}$.
20. 390625.	27. .056644.	34. $\frac{225}{1764}$.

Find to three decimal places the square root of

35. 2.	38. 3.5.	41. $\frac{3}{5}$.	44. $\frac{5}{7}$.
36. 3.	39. 6.6.	42. $\frac{2}{8}$.	45. $\frac{4}{15}$.
37. 5.	40. 0.9.	43. $\frac{7}{8}$.	46. $\frac{11}{20}$.

Cube Roots of Polynomials.

355. The process of extracting the cube root of a polynomial is readily derived from the formula

$$(a \pm b)^3 \equiv a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

This identity shows that $a \pm b$ is the cube root of

$$a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

It is observed that $\sqrt[3]{a^3}$, or a, is the first term of the cube root; and $\pm 3a^2b \div 3a^2$, or b, the second term of the root, $3a^2$ being the trial divisor; and since

$$\pm 3a^2b + 3ab^2 \pm b^3 = (3a^2 + 3ab + b^2)b,$$

$3a^2 + 3ab + b^2$ is a *complete divisor* of the last three terms of the polynomial, b being the other factor.

356. It is thus seen that the cube root of $a^3+3a^2b+3ab^2+b^3$ may be found by taking the cube root of a^3 for the first term of the root, and dividing $3a^2b$ by $3a^2$ for the second term of the root. Thus,

$$a^3+3a^2b+3ab^2+b^3 \mid a+b$$

$$(a)^3 = a^3$$

Trial divisor, $3a^2 \mid 3a^2b+3ab^2+b^3$

Complete divisor, $3a^2+3ab+b^2$

$$(3a^2+3ab+b^2)b = 3a^2b+3ab^2+b^3$$

1. What is the cube root of $x^6 + 9x^4y + 27x^2y^2 + 27y^3$?

$$x^6+9x^4y+27x^2y^2+27y^3 \mid x^2+3y$$

a^3 . . . . . $(x^2)^3 = x^6$

$3a^2$ (T. D.) . . . $3x^4 \mid 9x^4y+27x^2y^2+27y^3$

$3a^2+3ab+b^2$ (C.D.) $3x^4+9x^2y+9y^2$

$$(3x^4+9x^2y+9y^2)3y = 9x^4y+27x^2y^2+27y^3$$

357. A cube root having any number of terms may be found in like manner if it be observed that each successive trial divisor is *three times the square of the part of the root already found;* and that each successive divisor is completed *by adding to the trial divisor* (1) *three times the product of the root term last found multiplied by the part of the root before found, and* (2) *the square of the term last found.*

2. What is the cube root of

$$8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1?$$

$$\mid 2x^2-3x+1$$

$$8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$$

$(2x^2)^3 = 8x^6$

(T. D.) $3(2x^2)^2 = 12x^4 \mid -36x^5+66x^4-63x^3$

$(12x^4-18x^3+9x^2)(-3x) = -36x^5+54x^4-27x^3$

(T. D.) $3(2x^2-3x)^2 = 12x^4-36x^3+27x^2 \mid 12x^4-36x^3+33x^2-9x+1$

$(12x^4-36x^3+27x^2+6x^2-9x+1)1 = 12x^4-36x^3+33x^2-9x+1$

Find the cube root of

3. $x^6 + 6ax^4 + 12a^2x^2 + 8a^3$.
4. $8x^3 - 12ax^2 + 6a^2x - a^3$.
5. $125a^9 - 225a^6b^2 + 135a^3b^4 - 27b^6$.
6. $a^6 + 3a^5 + 6a^4 + 7a^3 + 6a^2 + 3a + 1$.
7. $x^6 + 6x^5 - 40x^3 + 96x - 64$.

8. $8x^6 - 12x^5 - 54x^4 + 59x^3 + 135x^2 - 75x - 125$.

9. $1 + 9x + 18x^2 - 27x^3 - 54x^4 + 81x^5 - 27x^6$.

10. $x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$.

11. $a^3 - 3a^2 + 9a - 13 + \frac{18}{a} - \frac{12}{a^2} + \frac{8}{a^3}$.

12. $27a^6 - 108a^5b + 90a^4b^2 + 80a^3b^3 - 60a^2b^4 - 48ab^5 - 8b^6$.

13. $8x^6 - 36x^5y + 66x^4y^2 - 63x^3y^3 + 33x^2y^4 - 9xy^5 + y^6$.

14. $27x^6 - 54ax^5 + 63a^2x^4 - 44a^3x^3 + 21a^4x^2 - 6a^5x + a^6$.

358. To extract the cube root of a polynomial,

Arrange the terms according to the powers of some letter.

Find the cube root of the first term, write it as the first term of the root, and subtract its cube from the polynomial.

Divide the first of the remaining terms by three times the square of the first term of the root (trial divisor), and write the quotient as the second term of the root.

To the trial divisor add three times the product of the first term multiplied by the second, and the square of the second; and then multiply the complete divisor thus formed by the second term of the root, and subtract the product from the remaining terms of the polynomial.

Proceed in like manner until all the terms of the polynomial are used.

Cube Roots of Numbers expressed by Figures.

359. The cubes of the smallest and the greatest integers expressed by one, two, and three figures, are as follows:

$1^3 = 1$	$10^3 = 1000$	$100^3 = 1000000$
$9^3 = 729$	$99^3 = 970299$	$999^3 = 997002999$

It is thus seen that the cube of an integer expressed by *one* figure contains *from one to three orders;* that the cube of an integer expressed by *two* figures contains *from four to six orders;* and that the cube of an integer expressed by *three* figures contains *from seven to nine orders.*

360. It follows, that the cube of an integer contains *three times as many orders as the integer, or three times as many orders less one or less two.*

361. The cubes of the smallest and the greatest numbers composed wholly of units, or tens, or hundreds, are as follows:

$1^3 = 1$	$10^3 = 1000$	$100^3 = 1000000$
$9^3 = 729$	$90^3 = 729000$	$900^3 = 729000000$

It is thus seen, that if a number be separated into periods of three orders each, beginning at the right, the *first* period will contain the cube of the *units* of its cube root; the *second* period, the cube of the *tens* of its cube root; the *third* period, the cube of the *hundreds* of its cube root; and so on.

362. It follows, that the cube root of an integer contains as many orders as there are periods in the integer, and *the cube root of the left-hand period of an integer is the left-hand term of its cube root.*

363. An integer may be separated into periods of three figures each by placing a dot over the first or units, fourth, seventh, etc., orders, thus, $48\dot{2}2\dot{8}54\dot{4}$; or by placing a dot between the periods, thus, 48'228'544.

Separate into periods and give the first figure of the cube root of

1. 42875.	3. 117649.	5. 274625.
2. 91125.	4. 185193.	6. 9405424.

364. The cube of a number may be found by the binomial formula $(a \pm b)^3 \equiv a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3$, as shown below.

7. What is the cube of 85?

$85 = 80 + 5.$

Let $a = 80$, and $b = 5$.

Since $(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$

$80^3 =$	512000
$3 \times 80^2 \times 5 =$	96000
$3 \times 80 \times 5^2 =$	6000
$5^3 =$	125

$(80 + 5)^3 = 80^3 + 3 \times 80^2 \times 5 + 3 \times 80 \times 5^2 + 5^3 = 614125$

365. It is thus seen that the cube of any number composed of tens and units is equal to (1) *the cube of the tens,* (2) *plus three times the product of the square of the tens multiplied by the units,* (3) *plus three times the product of the tens multiplied by the square of the units,* (4) *plus the cube of the units.*

Find thus the cube of

8. 82.	10. 77.	12. 104.
9. 68.	11. 56.	13. 125.

14. What is the cube root of 262144?

$$\begin{array}{lrrr}
 & a^3+(3a^2+3ab+b^2)b \;\ldots\; = & & 262144 \mid \overset{t}{6}\,\overset{u}{4} \\
 & a^3 \;\ldots\;\ldots\;\ldots\; 6^3 = & & 216 \\
\text{(T. D.)} & 3a^2 \;\ldots\; 3\times 60^2 = 10800 & \mid & 46144 \\
 & 3ab \;.\; 3\times 60\times 4 = 720 & & \\
 & b^2 \;\ldots\; 4^2 = 16 & & \\
 & (3a^2+3ab+b^2)b \;\ldots\; = 11536\times 4 = & & 46144
\end{array}$$

It is seen that the first *trial divisor* is $3\times 60^2 = 10800$, and that the *complete divisor* is 11536.

15. What is the cube root of 16387064?

$$\begin{array}{lrrr}
 & & & 16387064 \mid 254 \\
a^3 \;\ldots\;\ldots\;\ldots\; (2)^3 = & & & 8 \\
3a^2 \;\ldots\;\ldots\; 3\times 20^2 = 1200 & & \mid & 8387 \\
3ab \;\ldots\; 3\times 20\times 5 = 300 & & & \\
b^2 \;\ldots\;\ldots\; 5^2 = 25 & & & \\
(3a^2+3ab+b^2)b = 1525\times 5 = & & & 7625 \\
3a^2 \;\ldots\;\ldots\; 3\times 250^2 = 187500 & & \mid & 762064 \\
3ab \;\ldots\; 3\times 250\times 4 = 3000 & & & \\
b^2 \;\ldots\;\ldots\; 4^2 = 16 & & & \\
(3a^2+3ab+b^2)b = 190516\times 4 = & & & 762064
\end{array}$$

It should be noted, that, in finding the first two terms of the root (25), a in the formula denotes hundreds, and b tens (hundreds being considered tens, and tens units); but, in finding the third term (4), a denotes 25 tens, and b the units sought.

The process may be continued to any number of terms in the root by observing, when finding a new term, that a in the formula denotes *the part of the root already found.*

366. Since $.1^3 = .001$, $.01^3 = .000001$, etc., the cube of a decimal has three times as many decimal orders as the decimal; and hence a decimal is separated into periods of three orders each by beginning at the decimal point, and pointing off to the right, thus, $.015\dot{6}25\dot{.}$ If the last period does not contain three figures, decimal ciphers must be added; thus, .262500.

367. Since the cube of a fraction is obtained by raising each term to the third power (§ 315), the cube root of a fraction is found *by extracting the cube root of each of its terms.* If the denominator of a fraction is not a perfect cube, its approximate cube root may be most readily found by first changing the fraction to a decimal.

Find the cube root of

16. 42875.
17. 185193.
18. 3048625.
19. 48228544.
20. 34328125.
21. 27270901.
22. 74.088.
23. 97.336.
24. .097336.
25. 1953.125.
26. 67.419143.
27. 28.094464.
28. $\frac{216}{1728}$.
29. $\frac{2744}{8859}$.
30. $5\frac{104}{125}$.
31. $37\frac{1}{27}$.
32. $\frac{.625}{1.728}$.
33. $\frac{1.5}{.216}$.
34. $\frac{4}{.0625}$.

Find to three orders the cube root of

35. 2.
36. 9.
37. 10.
38. $\frac{2}{3}$.
39. $\frac{5}{6}$.
40. $\frac{1}{2}$.
41. $\frac{5}{64}$.
42. $\frac{1}{10}$.
43. $\frac{15}{216}$.

368. To extract the cube root of a number,

Begin at units, and separate the number into periods of three figures each.

Find the greatest cube in the left-hand period, and write its cube root as the first term of the required root.

Subtract the cube of the first term of the root from the left-hand period, and to the difference annex the second period for a dividend.

Divide this dividend by three times the square of the first term of the root, with two ciphers annexed (trial divisor), and write the quotient as the second term of the root.

To the trial divisor add three times the product of the first term multiplied by the second, and the square of the second; and then multiply the complete divisor thus formed by the second term of the root, subtract the product from the dividend, and to the difference annex the next period as another dividend.

Proceed in like manner until all the figures of the given number are used.

369. The fourth root of a number may be found by extracting the square root of its square root; and the sixth root, by extracting the cube root of its square root, or the square root of the cube root. The cube root of the square root is preferable in practice. Thus,

$$\sqrt[4]{625} = \sqrt{\sqrt{625}} = \sqrt{25} = 5; \quad \sqrt[6]{4096} = \sqrt[3]{\sqrt{4096}} = \sqrt[3]{64} = 4.$$

Find, as above, the fourth root of

44. $16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$.

45. $x^8 - 8x^7 + 16x^6 + 16x^5 - 56x^4 - 32x^3 + 64x^2 + 64x + 16$.

Find the sixth root of

46. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

47. $1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6$.

CHAPTER XIII.

RADICALS.

370. A **radical** is the indicated root of a number; as $\sqrt[3]{ab}$.

The term *radical* is also applied to expressions that contain a radical; as $3a\sqrt{ab}$, and $3a + \sqrt{ab}$ (§§ 375, 394).

371. If the indicated root can be exactly obtained, the radical is called **rational**; if the indicated root cannot be exactly obtained, the radical is called **irrational** or a **surd**. Thus, $\sqrt{25}$ and $\sqrt[3]{a^6}$ are rational, and $\sqrt{5}$ and $\sqrt[3]{a+b}$ are surds.

Roots may also be indicated by fractional exponents, as $5^{\frac{1}{2}}$, $(a+b)^{\frac{1}{3}}$ (§ 332); and roots thus indicated may be rational or irrational. Thus, $5^{\frac{1}{2}}$ and $(a+b)^{\frac{1}{3}}$ are surds.

Irrational numbers are also called *incommensurable*, since they have no common measure with unity.

372. It is to be observed that algebraic surds may become rational when numerical values are assigned to the letters. Thus, if we make $a = 6$, and $b = 3$, the surd $\sqrt{a+b}$ becomes rational. Surds that can be rationalized are said to be *surds in form.*

373. The **degree** of a radical is indicated by the index of the radical sign. Thus, $\sqrt{ab}$ (index 2 understood) is a radical of the *second* degree; $\sqrt[3]{a+b}$, a radical of the *third* degree; and so on.

Radicals of the second degree are also called *quadratic* radicals; and those of the third degree, *cubic* radicals.

The degree of a radical is also called its *order*. Since surds are a *class* of radicals, what is true of radicals generally is true of surds.

374. The **coefficient** of a radical is the factor placed before the radical part. Thus, in the expression $5\sqrt{a-b}$, 5 is the

coefficient of $\sqrt{a-b}$; and in the expression $7a\sqrt[3]{bx}$, $7a$ is the coefficient of $\sqrt[3]{bx}$.

375. When a radical contains no coefficient (except 1 understood), it is said to be *entire;* and when it contains a coefficient, it is said to be *mixed.* Thus, $\sqrt[3]{ax}$ and $\sqrt{a-x}$ are entire radicals, and $2\sqrt{5}$ and $5a\sqrt[3]{x-y}$ are mixed radicals.

The coefficient of a radical is called the *rational factor;* and the radical part, if a surd, the *irrational factor.*

376. **Similar radicals** are those which have the same index, and the same number under the radical sign; i.e., have their radical parts identical. Thus, $2\sqrt{3}$ and $a\sqrt{3}$ are similar; so, also, are $3\sqrt[3]{-2}$ and $2a\sqrt[3]{-2}$.

377. Since $\sqrt[m]{a}\times\sqrt[m]{b}=\sqrt[m]{ab}$, and, conversely, $\sqrt[m]{ab}=\sqrt[m]{a}\times\sqrt[m]{b}$, it follows (1) that *the product of the same roots of two factors equals the same root of their product;* and conversely (2) that *the root of any product equals the product of the same roots of its factors.*

378. This principle enables us to reduce a mixed radical to an entire radical, and an entire radical to a mixed radical; also to make other important reductions of radicals in degree and in form, as shown below.

REDUCTION OF RADICALS TO EQUIVALENT RADICALS OF DIFFERENT DEGREE.

379. Since $a=\sqrt{a^2}$, or $\sqrt[3]{a^3}$, or $\sqrt[m]{a^m}$, it follows that any rational number may be changed to an equivalent radical of any degree *by raising it to the power corresponding to the index of the radical, and placing the result under the radical sign.*

Thus, $ax=\sqrt[3]{a^3x^3}$; $\frac{a}{3}=\sqrt[4]{\frac{a^4}{81}}$; and $a-b=\sqrt[m]{(a-b)^m}$.

380. In like manner a radical of any degree may be changed to an equivalent radical of a higher degree *by multiplying the index of the radical, and the exponent of each factor under the radical sign, by the same number.* Thus,

$$\sqrt{5} = \sqrt[6]{5^3} = \sqrt[6]{125}; \quad \sqrt[3]{ab} = \sqrt[6]{a^2b^2}; \text{ and } \sqrt{a+b} = \sqrt[2n]{(a+b)^n}.$$

381. Conversely, any radical may be changed to an equivalent radical of a lower degree *by dividing the index, and the exponent of each factor under the radical sign, by the same number.* Thus,

$$\sqrt[6]{a^2} = \sqrt[3]{a}; \quad \sqrt[4]{(a+b)^2} = \sqrt{a+b}; \text{ and } \sqrt[6]{a^6x^9} = \sqrt{a^2x^3}.$$

Reduce the following radicals to equivalent radicals of the indicated degree:

1. ab to third.
2. $x - y$ to second.
3. $\frac{3}{x^2}$ to fifth.
4. $\sqrt{ax}$ to sixth.
5. $\sqrt[3]{x+y}$ to sixth.
6. $\sqrt{(a-x)^n}$ to nth.
7. $\sqrt[n]{ab}$ to mth.
8. $\sqrt[6]{a^3b^6}$ to second.
9. $\sqrt[2n]{(x-y)^2}$ to nth.
10. $\sqrt{\frac{9}{(x+y)^2}}$ to second.
11. $\frac{a}{x}$ to fourth.
12. $\sqrt{\frac{x}{x-y}}$ to sixth.
13. $\sqrt[mn]{a^m c^m d^{2m}}$ to nth.
14. $\sqrt{\frac{(a+b)^3}{a^3b^3}}$ to second.

382. Radicals of different degrees may be reduced to equivalent radicals of any degree which is a *multiple* of their indices. Thus, $\sqrt{5}$ and $\sqrt[3]{7}$ may each be changed to the sixth degree, as above; and $\sqrt{ac}$, $\sqrt[3]{a-b}$, and $\sqrt[4]{a+b}$ may each be changed to the twelfth degree.

383. This process is called the reduction of radicals to a *common index.* When the common index is the L. C. M. of the indices, it is a common index of the lowest degree.

Reduce to a common index of the lowest degree

15. $\sqrt{5}$, $\sqrt[3]{4}$, and $\sqrt[4]{10}$.

16. $\sqrt[3]{x^2}$, $\sqrt[4]{xy}$, and $\sqrt[6]{x^3y^3}$.

17. $\sqrt[4]{a}$, $\sqrt[6]{2b}$, and $\sqrt[12]{a^3b^6}$.

18. $\sqrt[3]{\frac{m}{n}}$, $\sqrt[4]{\frac{2}{n}}$, and $\sqrt[6]{\frac{m}{3}}$.

19. $\sqrt{\frac{1}{a}}$, $\sqrt[4]{\frac{1}{b^2}}$, and $\sqrt[8]{\frac{1}{c^4}}$.

20. x, $\sqrt[3]{x^2y}$, and $\sqrt[6]{x^3y^2}$.

21. $\sqrt[4]{x-y}$ and $\sqrt[6]{x+y}$.

22. $a-b$, $\sqrt[4]{(a-b)^2}$, and $\sqrt[6]{(a-b)^3}$.

23. $\sqrt{a^2b}$, $\sqrt[5]{\frac{b}{a^2}}$, $\sqrt[10]{\frac{a^2b^2}{(a+b)^5}}$.

REDUCTION OF RADICALS TO SIMPLEST FORM.

384. A radical is in its *simplest form* when its radical part is integral, and contains no factor which is a perfect power of the same degree as the radical. Thus, $\sqrt{a^2-b^2}$ and $3\sqrt[3]{a^2b}$ are radicals in their simplest form.

385. When the number under the radical sign contains a factor which is a perfect power of the same degree as the radical, the radical may be reduced to its simplest form *by removing such factor from under the radical sign, and making its proper root the coefficient, or a factor of the coefficient.* Thus,

$$\sqrt{25\,a^2x} = \sqrt{25\,a^2 \times x} = \sqrt{25\,a^2} \times \sqrt{x} = 5a\sqrt{x};$$

$$3\sqrt[3]{a^3b^2} = 3\sqrt[3]{a^3} \times \sqrt[3]{b^2} = 3a\sqrt[3]{b^2}.$$

Reduce to simplest form

1. $\sqrt{75}$.

2. $\sqrt[3]{320}$.

3. $2\sqrt[4]{80}$.

4. $\frac{3}{4}\sqrt[3]{192}$.

5. $\sqrt{32\,a^3b^4}$.

6. $3a\sqrt[3]{625\,b^4y^2}$.

7. $\sqrt{125\,a^3 - 50\,a^2b}$.
8. $\sqrt[3]{16\,x^6y^2 - 24\,x^2y^6}$.
9. $\sqrt{(a+b)(a^2-b^2)}$.
10. $\sqrt{(x-y)(x^3-y^3)}$.
11. $\sqrt[3]{(x+y)^2(x^2-y^2)}$.
12. $(x+y)\sqrt{x^3 - 2\,x^2y + xy^2}$.
13. $\frac{2}{5}(a+5)\sqrt{3\,a^2b^2 - 30\,ab^2 + 75\,b^2}$.
14. $5\,a\sqrt{x - 2\,xy + xy^2}$.
15. $(x+y)\sqrt{x^4 - x^2y^2}$.

386. When a radical is fractional, it may be reduced to its simplest form *by multiplying both terms of the fraction by such a number as will make the denominator a perfect power of the degree of the radical, and then proceeding as above.* Thus,

$$\sqrt{\frac{2}{5}} = \sqrt{\frac{2\times 5}{5^2}} = \sqrt{\frac{1}{5^2}\times 10} = \frac{1}{5}\sqrt{10};$$

$$\sqrt[3]{\frac{a}{b}} = \sqrt[3]{\frac{ab^2}{b^3}} = \sqrt[3]{\frac{1}{b^3}\times ab^2} = \frac{1}{b}\sqrt[3]{ab^2}.$$

Reduce to simplest form

16. $\sqrt{\frac{7}{12}}$.
17. $\sqrt[3]{\frac{5}{4}}$.
18. $\sqrt[4]{\frac{2}{9}}$.
19. $3\sqrt[3]{\frac{2}{8}}$.
20. $10\sqrt{\frac{3}{5}\,a^2x}$.
21. $10\,a^2b\sqrt[3]{\frac{3}{25\,a^2b^2}}$.
22. $\frac{a}{b}\sqrt{\frac{b^2x}{a^3}}$.
23. $\frac{5}{a-b}\sqrt{\frac{a-b}{5(a+b)}}$.
24. $6\,a\sqrt{\frac{a+b}{3(a-b)}}$.
25. $(x-y)\sqrt{\frac{x+y}{(x-y)^3}}$.
26. $x^2 - y^2\sqrt[3]{\frac{1}{x^2 - 2\,xy + y^2}}$.
27. $\frac{x-1}{3}\sqrt[3]{\frac{27\,(x+1)}{(x-1)^2}}$.

387. When the quantity under the radical sign is a perfect power whose exponent is a factor of the index, a radical may be simplified *by dividing both the index and the exponent by the common factor.* Thus,

$\sqrt[6]{a^4b^2} = \sqrt[3]{a^2b}$; $\sqrt[4]{(a+b)^2} = \sqrt{a+b}$; and $\sqrt[3]{(a-b)^6} = (a-b)^2$.

Reduce to simplest form

28. $\sqrt[4]{100}$.

29. $\sqrt[6]{125\,a^3x^3}$.

30. $\sqrt[4]{(a^2-b^2)^2}$.

31. $\sqrt[4]{(a^2-b^2)^2(a+b)^4}$.

32. $\sqrt[9]{(a+b)^3(a-b)^6}$.

33. $\sqrt[6]{4\,a^2-24\,ab+36\,b^2}$.

34. $\sqrt[10]{x^5y^{10}(x-y)^5}$.

35. $\sqrt[8]{a^4b^2(a+b)^2}$.

INTRODUCTION OF COEFFICIENTS UNDER THE RADICAL SIGN.

388. The coefficient of a radical may be placed under the radical sign *by raising it to the power corresponding to the index, and introducing the result under the sign as a factor.* Thus,

$$5\sqrt{ax}=\sqrt{25}\times\sqrt{ax}=\sqrt{25\,ax};$$

$$5\,a\sqrt{xy}=\sqrt{25\,a^2}\times\sqrt{xy}=\sqrt{25\,a^2xy}.$$

It will be observed that this is the converse of § 385. Thus, $\sqrt{4\,a^2x}=2\,a\sqrt{x}$, and, conversely, $2\,a\sqrt{x}=\sqrt{4\,a^2x}$.

Introduce under the radical sign the coefficient of

1. $2\sqrt[3]{ax}$.

2. $-3\,a\sqrt{10\,b}$.

3. $\frac{1}{2}\sqrt{5\,xy}$.

4. $\frac{a}{2}\sqrt{ay^2}$.

5. $2\,ax\sqrt[3]{\frac{1}{4\,a^2x^2}}$.

6. $3\,x\sqrt[4]{\frac{y}{9\,x^2}}$.

7. $(m-1)\sqrt{\frac{m+1}{m-1}}$.

8. $\frac{m+n}{m-n}\sqrt{\frac{m-n}{m+n}}$.

9. $\frac{1}{m-n}\sqrt[3]{m^3-n^3}$.

10. $(a+b)\sqrt{\frac{a-b}{a^3+b^3}}$.

11. $(a-b)\sqrt[3]{\frac{1}{(a-b)^2}}$.

12. $(1+a)\sqrt{\frac{2}{1+a}-1}$.

13. $(a-b)\sqrt{\frac{a}{a-b}-\frac{a+b}{a-b}}$.

ADDITION AND SUBTRACTION OF RADICALS.

389. Since the coefficient of a radical denotes the number of times it is taken, it is evident that the sum or difference of two similar radicals is found *by prefixing the sum or difference of their coefficients to the common radical part.* Thus,

$7\sqrt{ab} + 5\sqrt{ab} = 12\sqrt{ab}$; and $7\sqrt{a-b} - 5\sqrt{a-b} = 2\sqrt{a-b}$.

390. If the radicals to be added or subtracted are dissimilar, they must first be changed to similar radicals (§ 376). When this cannot be done, their sum or difference can only be indicated by the proper sign.

1. Add $7\sqrt{8}$, $3\sqrt{32}$, and $-8\sqrt{\frac{1}{2}}$.

$$\begin{aligned} 7\sqrt{8} = 7\sqrt{4\times 2} &= 14\sqrt{2} \\ 3\sqrt{32} = 3\sqrt{16\times 2} &= 12\sqrt{2} \\ -8\sqrt{\tfrac{1}{2}} = -8\sqrt{\tfrac{1}{4}\times 2} &= -4\sqrt{2} \\ \text{Sum} &= 22\sqrt{2} \end{aligned}$$

2. Simplify $\sqrt{8(a-b)} + 4\sqrt{18(a-b)} - 12\sqrt{\frac{1}{8}(a-b)}$.

$$\begin{aligned} \sqrt{8\,(a-b)} = \sqrt{4\times 2\,(a-b)} &= 2\sqrt{2\,(a-b)} \\ 4\sqrt{18\,(a-b)} = 4\sqrt{9\times 2\,(a-b)} &= 12\sqrt{2\,(a-b)} \\ -12\sqrt{\tfrac{1}{8}\,(a-b)} = -12\sqrt{\tfrac{2}{16}\,(a-b)} &= -3\sqrt{2\,(a-b)} \\ \text{Value} &= 11\sqrt{2\,(a-b)} \end{aligned}$$

If the given radicals are of different degrees, they must first be reduced to the same degree (§§ 380, 381).

Simplify

3. $\sqrt{18} + \sqrt{32}$.
4. $\sqrt{54} + 2\sqrt{294}$.
5. $12\sqrt{20} - 3\sqrt{45}$.
6. $4\sqrt{\frac{6}{5}} - 6\sqrt{\frac{40}{27}}$.
7. $16\sqrt{\frac{125}{12}} - 4\sqrt{\frac{3}{5}} - \sqrt{\frac{15}{16}}$.
8. $2\sqrt[3]{250} + 3\sqrt[3]{54}$.
9. $\sqrt[3]{1080} - \sqrt[3]{40} + \sqrt[3]{135}$.
10. $3\sqrt{80} + 6\sqrt{45} - 2\sqrt{125}$.
11. $\sqrt{\frac{1}{2}} - 2\sqrt{\frac{1}{8}} - \sqrt{\frac{4}{18}}$.
12. $\sqrt{2} + \sqrt{8} - \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}}$.

13. $4\sqrt{\frac{3}{4}}+\frac{1}{2}\sqrt{\frac{1}{3}}-3\sqrt{\frac{3}{16}}$.

14. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{27}}$.

15. $\sqrt[4]{48}+\sqrt[4]{\frac{1}{27}}+8\sqrt[4]{\frac{3}{16}}$.

16. $\sqrt[3]{270}-5\sqrt[3]{80}+\sqrt[3]{640}$.

17. $2\sqrt[3]{\frac{1}{2}}+4\sqrt[6]{\frac{1}{4}}-\sqrt[9]{\frac{1}{8}}$.

18. $\sqrt{\frac{1}{a}}+\sqrt[6]{\frac{1}{a^3}}-\sqrt[4]{\frac{1}{a^2}}$.

19. $\sqrt{\frac{3}{a^2}}+\sqrt{\frac{3}{b^2}}-\sqrt{\frac{3}{x^2}}$.

20. $\frac{2}{3}\sqrt{18}-\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{8}}+3\sqrt{\frac{1}{18}}$.

21. $3a\sqrt{x^3y^2}-5a\sqrt{xy^4}$.

22. $5a^2\sqrt{ax^3}-5ax\sqrt{a^3x}$.

23. $3a^2\sqrt[3]{a^5x^2}-3x^2\sqrt[3]{a^2x^5}$.

24. $2c\sqrt{\frac{x}{a^2y}}+30\sqrt{\frac{x^3}{y^3}}$.

25. $3\sqrt[3]{c}-5\sqrt[6]{c^2}+2\sqrt[9]{c^3}$.

26. $5\sqrt{(a-x)^3}-2ax\sqrt{a-x}$.

391. To find the sum or difference of radicals,

Reduce the radicals, if necessary, to similar radicals, and then prefix the sum or difference of the coefficients to the radical part.

If the radicals cannot be made similar, indicate their sum or difference by the appropriate sign.

MULTIPLICATION OF RADICALS.

392. Since $a\sqrt[n]{a}\times b\sqrt[n]{b}=a\times b\times\sqrt[n]{a}\times\sqrt[n]{b}=ab\sqrt[n]{ab}$, it follows that the product of two or more radicals of the same degree equals the product of their coefficients prefixed to the product of their radical factors.

If the radicals are of different degrees, they must first be reduced to equivalent radicals of the same degree.

1. What is the product of $6\sqrt{8}$ and $7\sqrt{6}$?

$$6\sqrt{8}\times 7\sqrt{6}=6\times 7\times\sqrt{8\times 6}=42\sqrt{16\times 3}=168\sqrt{3}.$$

2. What is the product of $3\sqrt{2}$ and $5\sqrt[3]{4}$?

$$3\sqrt{2}=3\sqrt[6]{8};\quad 5\sqrt[3]{4}=5\sqrt[6]{16}.$$

$$\therefore\ 3\sqrt{2}\times 5\sqrt[3]{4}=3\sqrt[6]{8}\times 5\sqrt[6]{16}=3\times 5\sqrt[6]{8\times 16}=15\sqrt[6]{64\times 2}=30\sqrt[6]{2}.$$

3. What is the product of $\sqrt[m]{a}$ and $\sqrt[n]{b}$?

$$\sqrt[m]{a} = \sqrt[mn]{a^n}; \quad \sqrt[n]{b} = \sqrt[mn]{b^m}.$$

$$\therefore \sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[mn]{a^n} \times \sqrt[mn]{b^m} = \sqrt[mn]{a^n b^m}.$$

Simplify

4. $6\sqrt{8} \times 7\sqrt{2}$.
5. $5\sqrt{10} \times 2\sqrt{15}$.
6. $\frac{1}{2}\sqrt[3]{12} \times \frac{2}{3}\sqrt[3]{2}$.
7. $\frac{1}{2}\sqrt[4]{8} \times \frac{1}{3}\sqrt[4]{160}$.
8. $2\sqrt{\frac{3}{5}} \times 3\sqrt{\frac{5}{8}}$.
9. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{3}{4}} \times \sqrt{\frac{5}{6}}$.
10. $4\sqrt[3]{3} \times 5\sqrt[3]{9} \times 2\sqrt[3]{4}$.
11. $12\sqrt[3]{\frac{2}{5}} \times 2\sqrt[3]{\frac{5}{6}} \times 3\sqrt[3]{\frac{1}{4}}$.
12. $\sqrt{15} \times \sqrt[3]{10}$.
13. $2\sqrt{3} \times 3\sqrt[3]{2} \times \sqrt[6]{\frac{1}{2}}$.
14. $2\sqrt[3]{3} \times 2\sqrt{4} \times 3\sqrt[3]{2}$.
15. $\sqrt{3} \times \sqrt[3]{2} \times \sqrt[6]{\frac{1}{6}}$.
16. $2\sqrt[3]{\frac{1}{3}} \times 3\sqrt{\frac{1}{2}} \times \sqrt[4]{\frac{2}{3}}$.
17. $4\sqrt{a^2b} \times 3\sqrt[4]{ab^2}$.
18. $3\sqrt{ax} \times 2\sqrt{5xy}$.
19. $3\sqrt{ax^3} \times 2\sqrt{5ax} \times 3\sqrt{x}$.
20. $2\sqrt{\frac{ax^5}{y^3}} \times 3\sqrt{\frac{a^3y^5}{x^3}}$.
21. $2a\sqrt{\frac{xy^3}{5z}} \times 3x\sqrt{\frac{3z^3}{8x^3y}}$.
22. $3a\sqrt{b^3} \times 2b\sqrt[3]{a^2}$.
23. $3\sqrt{\frac{a+x}{a-x}} \times 4\sqrt[3]{a-x}$.
24. $a\sqrt{x^3} \times b\sqrt[3]{x^2} \times c\sqrt[6]{x}$.
25. $\sqrt{3ab^3} \times \sqrt[3]{4a^2b} \times \sqrt[6]{2a^4b^3}$.
26. $3a\sqrt{x^2-y^2} \times 5b\sqrt[3]{\frac{1}{x^2-y^2}}$.
27. $\sqrt[3]{5} \times \sqrt[3]{-25} \times \sqrt[3]{10}$.

393. To find the product of two or more radicals,

Reduce the given radicals, if necessary, to equivalent radicals of the same degree; then prefix the product of the coefficients to the product of the radical factors, and reduce the resulting radical to its simplest form.

394. A **compound radical** is a polynomial that has one or more radical terms; as, $2\sqrt{a} + 3\sqrt{b}$, $a - \sqrt{3b}$, $\sqrt{a} - \sqrt{a-b}$, and $\sqrt{a+b} - \sqrt{a-b}$.

28. Multiply $x+2\sqrt{3}$ by $x-\sqrt{3}$.

$$\begin{array}{r} x+2\sqrt{3} \\ x-\sqrt{3} \\ \hline x^2+2x\sqrt{3} \\ \quad -x\sqrt{3}-2\sqrt{3^2} \\ \hline x^2+x\sqrt{3}-6 \end{array}$$

The product of two binomial radicals may be written directly as in § 165.

Multiply

29. $\sqrt{x}-3$ by $\sqrt{x}+2$.
30. $2\sqrt{x}-4\sqrt{3}$ by $3\sqrt{x}+2\sqrt{3}$.
31. $2\sqrt{5}+3\sqrt{2}$ by $3\sqrt{5}-4\sqrt{2}$.
32. $2\sqrt{a}-3\sqrt{b}$ by $\sqrt{a}+\sqrt{b}$.
33. $4\sqrt{a}+\sqrt{b}$ by $4\sqrt{a}+\sqrt{b}$.
34. $x+\sqrt{xy}+y$ by $\sqrt{x}-\sqrt{y}$.
35. $a+a\sqrt{3}+1$ by $a\sqrt{3}-1$.
36. $\sqrt[3]{3}+2\sqrt[3]{2}$ by $2\sqrt[3]{3}+\sqrt[3]{2}$.
37. $2\sqrt{\frac{2}{3}}+3\sqrt{\frac{1}{2}}+1$ by $3\sqrt{\frac{1}{3}}-2\sqrt{\frac{1}{2}}$.

395. When two binomials involving radicals differ only in the sign of a radical term, they are said to be **conjugate**; and their sum and product are rational. Thus,

$(a+\sqrt{b})(a-\sqrt{b})=a^2-b$, and $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$.

Multiply

38. $2\sqrt{a}-\sqrt{b}$ by $2\sqrt{a}+\sqrt{b}$.
39. $\sqrt{x}+2\sqrt{y}$ by $\sqrt{x}-2\sqrt{y}$.
40. $3\sqrt{5}+2\sqrt{3}$ by $3\sqrt{5}-2\sqrt{3}$.
41. $\frac{1}{2}\sqrt{x}-\sqrt{12}$ by $\frac{1}{2}\sqrt{x}+\sqrt{12}$.
42. $\sqrt{x-\sqrt{21}}$ by $\sqrt{x+\sqrt{21}}$.
43. $1+\sqrt{x+1}$ by $1-\sqrt{x+1}$.
44. $\sqrt{a+x}-\sqrt{a-x}$ by $\sqrt{a+x}+\sqrt{a-x}$.

DIVISION OF RADICALS.

396. Since $a\sqrt[n]{x} \times b\sqrt[n]{y} = ab\sqrt[n]{xy}$, $ab\sqrt[n]{xy} \div b\sqrt[n]{y} = a\sqrt[n]{x}$; and hence the quotient of two radicals of the same degree equals the quotient of their coefficients prefixed to the quotient of their radical factors.

If the radicals are of different degrees, they must be reduced to radicals of the same degree before dividing.

1. Divide $12\sqrt{75}$ by $5\sqrt{3}$.

$$12\sqrt{75} \div 5\sqrt{3} = \tfrac{12}{5}\sqrt{\tfrac{75}{3}} = \tfrac{12}{5}\sqrt{25} = 12.$$

2. Divide $\sqrt{\frac{1}{2}}$ by $\sqrt[3]{\frac{1}{3}}$.

$$\sqrt{\tfrac{1}{2}} = \sqrt[6]{\tfrac{1}{8}}; \quad \sqrt[3]{\tfrac{1}{3}} = \sqrt[6]{\tfrac{1}{9}}.$$

$$\sqrt[6]{\tfrac{1}{8}} \div \sqrt[6]{\tfrac{1}{9}} = \sqrt[6]{\tfrac{1}{8} \div \tfrac{1}{9}} = \sqrt[6]{\tfrac{9}{8}} = \sqrt[6]{\tfrac{144}{64}} = \tfrac{1}{2}\sqrt[6]{72}.$$

Divide

3. $3\sqrt{108}$ by $3\sqrt{6}$.
4. $3\sqrt{8}$ by $5\sqrt{27}$.
5. $3\sqrt{6}$ by $\sqrt{8}$.
6. $3\sqrt{\frac{1}{2}}$ by $2\sqrt{\frac{2}{3}}$.
7. $12\sqrt{\frac{3}{5}}$ by $3\sqrt{\frac{27}{10}}$.
8. $\frac{3}{7}\sqrt[3]{\frac{2}{8}}$ by $\frac{5}{21}\sqrt[3]{\frac{9}{4}}$.
9. $6\sqrt{8} - 6\sqrt{2}$ by $3\sqrt{2}$.
10. $\sqrt{4}$ by $\sqrt[3]{6}$.
11. $\sqrt[3]{12}$ by $\sqrt{8}$.
12. $\sqrt{3\frac{1}{3}}$ by $\sqrt[3]{2\frac{1}{2}}$.
13. $\sqrt[3]{25}$ by $\sqrt{2\frac{1}{2}}$.
14. $2\sqrt{5}$ by $\sqrt[3]{5}$.
15. $15\sqrt{a^3b^3}$ by $3\sqrt{ab^3}$.
16. $2a\sqrt{a^3x}$ by $5x\sqrt[3]{ax^3}$.
17. $5a\sqrt[3]{a^2x^5}$ by $10x\sqrt{ax^3}$.
18. $5\sqrt{x^3}$ by $4\sqrt[3]{x^5}$.
19. $3a^2\sqrt{a^5x}$ by $2a\sqrt[3]{a^3x^3}$.
20. $\sqrt{x^2-y^2}$ by $\sqrt{x-y}$.

397. To find the quotient of two radicals,

Reduce the given radicals, if necessary, to equivalent radicals of the same degree; then prefix the quotient of the coefficients to the quotient of the radical factors, and reduce the resulting radical to its simplest form.

Rationalizing the Divisor.

398. If the denominator of a fraction is a radical, the denominator may be made rational *by multiplying both terms of the fraction by such a number as will make the denominator a perfect power of the degree of the radical.*

Thus, $\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2}{3}\sqrt{3}$;

$$\frac{2}{\sqrt[3]{a}} = \frac{\times \sqrt[3]{a^2}}{\sqrt[3]{a} \times \sqrt[3]{a^2}} = \frac{2\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \frac{2\sqrt[3]{a^2}}{a}.$$

Reduce to equivalent fractions with rational denominators

1. $\frac{5}{\sqrt{3}}$.
2. $\frac{\sqrt{5}}{\sqrt{3}}$.
3. $\frac{10}{\sqrt{2}}$.
4. $\frac{\sqrt{21}}{\sqrt{12}}$.
5. $\frac{3\sqrt{8}}{\sqrt{6}}$.
6. $\frac{\sqrt{10}}{2\sqrt{5}}$.
7. $\frac{3}{\sqrt[3]{4}}$.
8. $\frac{5}{\sqrt{3xy}}$.
9. $\frac{1}{\sqrt[4]{8a^2}}$.
10. $\frac{5}{2\sqrt{6}}$.
11. $\frac{x}{\sqrt[3]{y^2}}$.
12. $\sqrt{\frac{a}{b}}$.

399. In like manner the quotient of two radicals may be found *by rationalizing the divisor*, and then reducing the quotient to its simplest form. Thus,

$$3\sqrt{6} \div \sqrt{5} = \frac{3\sqrt{6} \times \sqrt{5}}{(\sqrt{5})^2} = \frac{3\sqrt{30}}{5} = \frac{3}{5}\sqrt{30}.$$

400. When a divisor is a binomial of the form $\sqrt{a} \pm \sqrt{b}$, it may be rationalized by multiplying both dividend and divisor by the *conjugate* of the binomial divisor (§ 395). Thus,

$$\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3} \times (\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{3+\sqrt{6}}{3-2} = 3+\sqrt{6}.$$

Rationalize the divisor and simplify

13. $\sqrt{84} \div \sqrt{3}$.

14. $\sqrt{108} \div \sqrt{6}$.

15. $\sqrt{48} \div \sqrt{12}$.

16. $3\sqrt{8} \div \sqrt{6}$.

17. $\sqrt{15} \div 2\sqrt{5}$.

18. $(\sqrt{3} - \sqrt{2}) \div \sqrt{6}$.

19. $\sqrt{3} \div \sqrt{10}$.

20. $\dfrac{6}{5 - \sqrt{7}}$.

21. $\dfrac{1}{\sqrt{10} - 3}$.

22. $\dfrac{21}{\sqrt{10} + \sqrt{3}}$.

23. $\dfrac{\sqrt{8}}{\sqrt{5} + \sqrt{3}}$.

24. $\dfrac{3 + \sqrt{5}}{3 - \sqrt{5}}$.

25. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$.

26. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.

27. $\dfrac{2\sqrt{3} + 3\sqrt{2}}{3\sqrt{3} - 2\sqrt{2}}$.

28. $\dfrac{\sqrt{7}}{5\sqrt{10} + 2\sqrt{3}}$.

29. $\dfrac{2\sqrt{3} + 3\sqrt{2}}{3\sqrt{3} + 4\sqrt{2}}$.

30. $\dfrac{1}{1 - \sqrt{2} + \sqrt{3}}$ 31. $\dfrac{1}{1 + \sqrt{2} - \sqrt{3}}$. 32. $\dfrac{3}{2 + \sqrt{3} + \sqrt{5}}$.

SUGGESTION. In Example 30, multiply both terms by $(1 - \sqrt{2}) - \sqrt{3}$, and then multiply both terms of resulting fraction (simplified) by $-2\sqrt{2}$.

401. The above method of rationalizing the divisor has special value in finding the approximate numerical value of a quotient when the divisor is a surd, as shown below.

33. Find the approximate value of $4 \div \sqrt{3}$.

$$\frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3} = \frac{\sqrt{48}}{3} = \frac{6.928+}{3} = 2.309+.$$

34. Find the approximate value of $\dfrac{5}{4 - \sqrt{2}}$.

$$\frac{5}{4 - \sqrt{2}} = \frac{5(4 + \sqrt{2})}{(4 - \sqrt{2})(4 + \sqrt{2})} = \frac{20 + 5\sqrt{2}}{16 - 2} = \frac{20 + \sqrt{50}}{14} = 1.9336+.$$

Find to three decimal places the value of

35. $\frac{2}{\sqrt{8}}$.

36. $\frac{10}{\sqrt{3}}$.

37. $\frac{5}{\sqrt{12}}$.

38. $\frac{5}{\sqrt{27}}$.

39. $\frac{3}{\sqrt{2}-2}$.

40. $\frac{\sqrt{3}}{2+\sqrt{3}}$.

41. $\frac{2-\sqrt{3}}{3-\sqrt{2}}$.

42. $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

43. $\frac{5}{\sqrt{3}+\sqrt{2}}$.

44. $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}}$.

45. $\frac{3+\sqrt{5}}{3-\sqrt{5}}$.

46. $\frac{\sqrt{8}}{\sqrt{8}-\sqrt{2}}$.

INVOLUTION AND EVOLUTION OF RADICALS.

402. Radicals may be raised to any power by substituting fractional exponents for the radical signs, and then proceeding as in the involution of numbers with positive integral exponents (§ 319). The roots of radicals may be found in like manner. For other operations with fractional exponents see Chapter XIV.

1. Find the square of $4\sqrt[3]{a}$.

$$(4\sqrt[3]{a})^2 = (4\,a^{\frac{1}{3}})^2 = 16\,a^{\frac{2}{3}} = 16\sqrt[3]{a^2}.$$

2. Find the cube of $3\sqrt{ab^3}$.

$$(3\sqrt{ab^3})^3 = (3\,a^{\frac{1}{2}}b^{\frac{3}{2}})^3 = 27\,a^{\frac{3}{2}}b^{\frac{9}{2}} = 27\sqrt{a^3b^9}.$$

Raise to the indicated power

3. $(\sqrt[3]{8})^4$.

4. $(5\sqrt[4]{3})^2$.

5. $(\sqrt[6]{12})^3$.

6. $(\sqrt[12]{32})^4$.

7. $(\sqrt[3]{64})^3$.

8. $(\sqrt[3]{x^2})^4$.

9. $(\sqrt{ab^3})^6$.

10. $(2\sqrt[3]{a^4b^2})^4$.

11. $(\sqrt[3]{a^2x^3})^6$.

12. $(2\sqrt{a^3bx^5})^4$.

13. $(4\sqrt[3]{a-b})^4$.

14. $(x^m\sqrt[n]{y})^n$.

15. Extract the cube root of $8\sqrt{ax^3}$.

$$\sqrt[3]{8\sqrt{ax^3}} = (8\,a^{\frac{1}{2}}x^{\frac{3}{2}})^{\frac{1}{3}} = 2\,a^{\frac{1}{6}}x^{\frac{3}{6}} = 2\sqrt[6]{ax^3}.$$

16. $\sqrt[3]{\sqrt{64}}$.
17. $\sqrt[4]{\sqrt{81}}$.
18. $\sqrt[3]{125\sqrt{8}}$.
19. $\sqrt{25\sqrt[3]{6}}$.
20. $\sqrt[3]{\sqrt{a^3}}$.
21. $\sqrt[5]{x^2\sqrt{x}}$.
22. $\sqrt[3]{\sqrt[4]{a^3bx}}$.
23. $\sqrt{a^{-3}\sqrt{a}}$.
24. $\sqrt[n]{a\sqrt[3]{x}}$.

Square Roots of Binomial Surds.

403. A binomial one or both of whose terms are surds is called a **binomial surd.** Thus, $a+\sqrt{b}$, $\sqrt{a}-b$, and $\sqrt{a}\pm\sqrt{b}$ are binomial surds.

The finding of the square root of a binomial surd which is the sum of a rational number and a quadratic surd, as $a\pm\sqrt{b}$, is of sufficient utility to justify a brief treatment here.

404. Since $(\sqrt{5}\pm\sqrt{3})^2=5\pm2\sqrt{15}+3=8\pm2\sqrt{15}$, then, conversely, $\sqrt{8\pm2\sqrt{15}}=\sqrt{5}\pm\sqrt{3}$; and hence $8\pm2\sqrt{15}$ is a *perfect square.* It is thus seen that *some* binomial surds of the form $a\pm\sqrt{b}$ are perfect squares.

405. A formula for finding the square root of such binomial surds may be obtained as follows:

$$(\sqrt{a+\sqrt{b}}\pm\sqrt{a-\sqrt{b}})^2=2a\pm2\sqrt{a^2-b}.$$

$$\therefore\ \sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}=\sqrt{2a+2\sqrt{a^2-b}}, \qquad (1)$$

and
$$\sqrt{a+\sqrt{b}}-\sqrt{a-\sqrt{b}}=\sqrt{2a-2\sqrt{a^2-b}}. \qquad (2)$$

Adding (2) to (1), member to member, and dividing by 2, we have

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}; \qquad (3)$$

and subtracting (2) from (1), and dividing by 2, we have

$$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}. \qquad (4)$$

406. It is evident from Formulas (3) and (4) that the square root of $a \pm \sqrt{b}$ may be readily found if $a^2 - b$ is a perfect square. This may usually be determined by inspection, and the square root of the binomial is then found by substituting the values of a and $\sqrt{a^2 - b}$ in the proper formula. The method is not practicable when $a^2 - b$ is not a perfect square.

1. Find the square root of $6 + \sqrt{11}$ by formula.

$$\sqrt{a^2 - b} = \sqrt{6^2 - 11} = \sqrt{25} = 5,$$

whence $$\sqrt{6 + \sqrt{11}} = \sqrt{\frac{6+5}{2}} + \sqrt{\frac{6-5}{2}} = \sqrt{\frac{11}{2}} + \sqrt{\frac{1}{2}} = \tfrac{1}{2}\sqrt{22} + \tfrac{1}{2}\sqrt{2}.$$

2. Find the square root of $17 - 4\sqrt{15}$.

$$\sqrt{17 - 4\sqrt{15}} = \sqrt{17 - \sqrt{240}}; \quad \sqrt{17^2 - 240} = \sqrt{49} = 7;$$

whence $$\sqrt{17 - 4\sqrt{15}} = \sqrt{\frac{17+7}{2}} - \sqrt{\frac{17-7}{2}} = \sqrt{12} - \sqrt{5} = 2\sqrt{3} - \sqrt{5}.$$

Find the square root of

3. $6 + \sqrt{20}$.
4. $10 + \sqrt{19}$.
5. $7 + 4\sqrt{3}$.
6. $11 - \sqrt{21}$.
7. $21 + 4\sqrt{5}$.
8. $7 + 2\sqrt{10}$.
9. $11 - \sqrt{72}$.
10. $\frac{7}{12} + \sqrt{\frac{1}{6}}$.
11. $\frac{8}{5} - \frac{2}{5}\sqrt{7}$.
12. $\frac{7}{8} + \frac{1}{8}\sqrt{24}$.
13. $6a + 2a\sqrt{5}$.
14. $\dfrac{9}{5 + 2\sqrt{6}}$.
15. $\dfrac{121}{9 - \sqrt{17}}$.

407. The square root of a binomial surd may also be found by first so reducing the surd term that its coefficient is 2; then separating the rational term into two parts such that twice the product of their square roots equals the surd term; and then *connecting such square roots with the sign of the surd term* (§ 339). Thus

(1) $$\sqrt{6 + \sqrt{11}} = \sqrt{\tfrac{11}{2} + 2\sqrt{\tfrac{11}{4}} + \tfrac{1}{2}} = \sqrt{\tfrac{11}{2}} + \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{22} + \tfrac{1}{2}\sqrt{2}.$$

(2) $$\sqrt{17 - 4\sqrt{15}} = \sqrt{12 - 2\sqrt{60} + 5} = \sqrt{12} - \sqrt{5} = 2\sqrt{3} - 5.$$

Find by this method the square root of 3 to 15 above.

Another Method.

408. If $a \pm \sqrt{b} = c \pm \sqrt{d}$, and a and c are rational, and $\sqrt{b}$ and $\sqrt{d}$ are surds, then $a = c$, and $b = d$.

For, if possible, let $a = c + x$;
then, substituting, $c + x \pm \sqrt{b} = c \pm \sqrt{d}$;
whence $x \pm \sqrt{b} = \pm \sqrt{d}$.
Squaring, $x^2 \pm 2x\sqrt{b} + b = d$;
transposing, $x^2 + b - d = \mp 2x\sqrt{b}$,

which is impossible, since a rational number cannot equal a surd; and hence

$$a = c, \text{ and } \sqrt{b} = \sqrt{d}, \text{ or } b = d.$$

409. The square root of a binomial surd of the general form $a \pm \sqrt{b}$ may be found as follows:

Suppose $\sqrt{a} \pm \sqrt{b} = \sqrt{x} \pm \sqrt{y}$; (1)
squaring, $a \pm \sqrt{b} = x \pm 2\sqrt{xy} + y$.

Hence, by § 408, we have

$$x + y = a, \text{ and } 2\sqrt{xy} = \sqrt{b};$$
or $$y + y = a, \text{ and } 4xy = b. \quad (2)$$

The value of $\sqrt{a \pm \sqrt{b}}$ can now be found *by finding the values of* x *and* y *in equations* (2), *and substituting the same in* (1).

16. Find the square root of $15 + 2\sqrt{56}$.

Let $\sqrt{15 + 2\sqrt{56}} = \sqrt{x} + \sqrt{y}$;
squaring $15 + 2\sqrt{56} = x + 2\sqrt{xy} + y$;
whence $x + y = 15$, and $2\sqrt{xy} = 2\sqrt{56}$;
or $x + y = 15$, and $xy = 56$. $\therefore x = 7$; $y = 8$.
Hence $\sqrt{15 + 2\sqrt{56}} = \sqrt{7} + \sqrt{8}$.

Find the square root of

17. $7 + 2\sqrt{10}$.
18. $6 + \sqrt{20}$.
19. $11 + 2\sqrt{30}$.
20. $12 - 6\sqrt{3}$.
21. $16 + 2\sqrt{55}$.
22. $5 - 2\sqrt{6}$.

EQUATIONS INVOLVING RADICALS.

410. An equation in which the unknown quantity occurs under the radical sign is called a **radical equation.**

411. The more common methods of solving radical equations are indicated below.

1. Solve the equation $\sqrt{x^2-15}+x=5.$

Transposing x, $\sqrt{x^2-15}=5-x;$
squaring, $x^2-15=25-10x+x^2;$
transposing, $10x=40;$
whence $x=4.$

2. Solve the equation $\sqrt{x+7}=7-\sqrt{x}.$

Squaring, $x+7=49-14\sqrt{x}+x;$
transposing, $14\sqrt{x}=42;$
dividing by 14, $\sqrt{x}=3;$
squaring, $x=9.$

3. Solve the equation $\sqrt{x-a}=\sqrt[4]{x^2+ax}.$

Raising $\sqrt{x-a}$ to 4th degree, $\sqrt[4]{(x-a)^2}=\sqrt[4]{x^2+ax};$
raising to 4th power, $x^2-2ax+a^2=x^2+ax;$
transposing and changing sign, $3ax=a^2;$
whence $x=\frac{a^2}{3a}=\frac{a}{3}$

Solve

4. $\sqrt{x+4}=6.$
5. $\sqrt{7x-38}=5.$
6. $\sqrt{x^2-16}=x-2.$
7. $\sqrt{x}-1=\sqrt{x-9}.$
8. $\sqrt{x+3}=\sqrt{3x-1}.$
9. $5\sqrt{x-9}=3\sqrt{x+5}.$
10. $\sqrt[3]{4x+3}=3.$
11. $\sqrt{5x+4}=\sqrt{3x}+2.$
12. $\sqrt{4x-19}=2\sqrt{x}-1.$
13. $\sqrt{x+5}+\sqrt{x}=5.$
14. $\sqrt{x+3}=\frac{x-1}{\sqrt{x-3}}.$
15. $\sqrt{x+12}=2+\sqrt{x}.$
16. $\sqrt{4+\sqrt{x-2}}=3.$

17. $\sqrt{x+3a}=3\sqrt{a}$.

18. $\sqrt{x^2+a^2}=b-x$.

19. $\sqrt{x+a^2}=\sqrt{x}+a$.

20. $\sqrt{x}-\sqrt{2}=\sqrt{x-2}$.

21. $\sqrt{x+4ab}=\sqrt{x}+2a$.

22. $2\sqrt{x}-\sqrt{a}=2\sqrt{x-a}$.

23. $\sqrt{x+a^2}-\sqrt{x-a^2}=\sqrt{2b}$.

24. $\sqrt{x+a}=\sqrt{a}+\sqrt{x-a}$.

25. $\sqrt{x+2}+\sqrt{x-2}=\dfrac{4}{\sqrt{x+2}}$.

26. $a+x=\sqrt{x^2+5ax-2a^2}$.

27. $\dfrac{a^2}{\sqrt{a^2-1}}-x=\dfrac{1}{\sqrt{a^2-1}}$.

28. $\dfrac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=b$.

29. $\sqrt{x^2-a^2x}=\sqrt[4]{x^4+b^4x^2}$.

30. $\sqrt{x+2}=\sqrt{1-\sqrt{x^2+21}}$.

31. $\dfrac{\sqrt{x}-2}{1}+3=\dfrac{x+8}{\sqrt{x}+2}$.

MISCELLANEOUS EXERCISES.

Simplify

1. $\frac{3}{4}y\sqrt{x^3y}-\frac{1}{3}ax\sqrt{x^5y}$.

2. $\dfrac{x-y}{x+y}\sqrt{\dfrac{x+y}{x-y}}+\sqrt{\dfrac{9(x^2-y^2)}{(x+y)^2}}$.

3. $\sqrt{ax^2-4a^2x+4a^3}+\sqrt{16a^3}$.

4. $(\frac{1}{2}-\sqrt[3]{\frac{1}{3}})\times\sqrt{\frac{2}{3}}$.

5. $\sqrt{\dfrac{a}{2}}\times\sqrt[3]{\dfrac{a}{3}}\times\sqrt[6]{\dfrac{a}{6}}$.

6. $\sqrt{a-\sqrt{b}}\times\sqrt{a+\sqrt{b}}$.

7. $\dfrac{x+y}{x+\sqrt{y}}$.

8. $\dfrac{\sqrt{ab}-ax}{bx-\sqrt{ab}}$.

9. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$.

10. $\dfrac{\sqrt{x-y}-\sqrt{x+y}}{\sqrt{x-y}+\sqrt{x+y}}$.

11. $\dfrac{\sqrt{x-4}-5}{\sqrt{x-4}+5}$.

12. $(x+y)\sqrt{\dfrac{x-y}{x+y}}$.

13. $\sqrt{11+4\sqrt{6}}$.

14. $\sqrt{16+5\sqrt{7}}$.

15. $\sqrt{9\sqrt{5}+21}$.

16. $\sqrt{21-4\sqrt{5}}$.

17. $\dfrac{3}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

18. $\dfrac{2}{\sqrt{2}+\sqrt{3}-1}$.

19. $\dfrac{1}{\sqrt{5}-\sqrt{2}}+\dfrac{1}{\sqrt{5}+\sqrt{2}}$.

20. $\dfrac{2}{\sqrt{7}+1}-\dfrac{1}{\sqrt{7}-1}$.

21. $\sqrt{6} \times \sqrt[3]{12} \times \sqrt[4]{8} \times \sqrt[6]{54} \times \sqrt[12]{100}$.

22. $(x+\sqrt{x^2}+\sqrt{y^2})(x-\sqrt{x^2}-\sqrt{y^2})$.

23. $(a^2+\sqrt{a}+\sqrt{b})(a^2-\sqrt{a}-\sqrt{b})$.

24. $\sqrt{a^2-b^2} \div (\sqrt{a-b} \times \sqrt{a-b})$

Find the value of x in

25. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}$.

26. $\sqrt{\dfrac{a^2+bx}{x}}-\sqrt{\dfrac{a^2-bx}{x}}=c$.

27. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=c$.

28. $\sqrt{a-x}=\dfrac{a}{\sqrt{a-x}}-x$.

29. $\sqrt{x-a}-\sqrt{x-b}=\sqrt{b}-\sqrt{a}$.

30. $2\sqrt{x}-\sqrt{4x-11}=1$.

IMAGINARY NUMBERS.

412. Since $(\pm a)^2=+a^2$, $(\pm a)^4=+a^4$, and $(\pm a)^{2n}=+a^{2n}$ (§ 316), it follows that every even power of any number is *positive;* and hence the even root of a negative number is impossible, and hence such a root can only be *indicated* by the proper sign.

413. The indicated even root of a negative number is called an **imaginary number.** Thus, $\sqrt{-a}$ and $\sqrt[4]{-a^2}$ are imaginary numbers.

In distinction from imaginary numbers, all other numbers are called **real numbers.**

An imaginary number is more properly called a *non-real* number, numbers being thus classified as real and non-real; but the term *imaginary number* is in almost universal use.

414. The indicated square root of a negative number is called a **quadratic imaginary number.**

It may be shown by the methods of higher algebra that every imaginary number may be expressed in the terms of a quadratic imaginary number, and hence we shall consider herein only *quadratic* imaginary numbers.

415. Since

$$\sqrt{-a^2}=\sqrt{a^2\times(-1)}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1},$$

and $$\sqrt{-a}=\sqrt{a}\times(-1)=\sqrt{a}\times\sqrt{-1}=a^{\frac{1}{2}}\sqrt{-1},$$

it appears that any imaginary square root may be changed to the form $a\sqrt{-1}$ or $a^{\frac{1}{2}}\sqrt{-1}$, in which a and $a^{\frac{1}{2}}$ are *real* numbers.

The radical $\sqrt{-1}$ is called the **imaginary unit.**

Reduce to the form $a\sqrt{-1}$

1. $\sqrt{-4}$.
2. $\sqrt{-25}$.
3. $\sqrt{-256}$.
4. $\sqrt{-16a^2}$.
5. $\sqrt{-x^8}$.
6. $\sqrt{\frac{-x^2}{4}}$.
7. $\sqrt{\frac{-9x^2}{16}}$.
8. $\sqrt{\frac{-x^4}{25}}$.
9. $\sqrt{-a^4x^2}$.
10. $\sqrt{-(a+b)^2}$.

416. An indicated square root of a number is squared by simply removing the radical sign, as $(\sqrt{a})^2=a$, and $(\sqrt{-a})^2=-a$; and hence the square of the imaginary unit $\sqrt{-1}$ is -1.

417. The successive powers of the imaginary unit are as follows:

$$(\sqrt{-1})^1 = \sqrt{-1};$$
$$(\sqrt{-1})^2 = -1;$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=-1\times\sqrt{-1}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{-1})^2=-1\times-1=+1;$$

and so on.

Hence, if n denotes any positive integer,

$$(\sqrt{-1})^{4n} = [(\sqrt{-1})^4]^n = (-1 \times -1)^n = +1;$$
$$\therefore (\sqrt{-1})^{4n+1} = +1\sqrt{-1} = \sqrt{-1};$$
$$\therefore (\sqrt{-1})^{4n+2} = +1 \times -1 = -1;$$
$$\therefore (\sqrt{-1})^{4n+3} = -1 \times \sqrt{-1} = -\sqrt{-1};$$
$$\therefore (\sqrt{-1})^{4n+4} = +1 \times +1 = +1.$$

418. It is thus seen that the first four powers of $\sqrt{-1}$ are respectively $\sqrt{-1}$, -1, $-\sqrt{-1}$, and $+1$; and that these numbers occur in the same order for the powers of $\sqrt{-1}$ whose exponents are $4n+1$ to $4n+4$ inclusive.

Addition and Subtraction.

419. Imaginary numbers may be added or subtracted in the same manner as other radicals (§ 389).

1. Add $\sqrt{-4}$ and $\sqrt{-25}$.

$$\sqrt{-4} = 2\sqrt{-1};\ \sqrt{-25} = 5\sqrt{-1};$$
$$2\sqrt{-1} + 5\sqrt{-1} = 7\sqrt{-1}.$$

2. Simplify $\sqrt{-144} - \sqrt{-49} - \sqrt{-16}$.

$$\begin{aligned} \sqrt{-144} &= 12\sqrt{-1} \\ -\sqrt{-49} &= -7\sqrt{-1} \\ -\sqrt{-16} &= -4\sqrt{-1} \\ \hline 12\sqrt{-1} - 7\sqrt{-1} - 4\sqrt{-1} &= \sqrt{-1} \end{aligned}$$

Simplify

3. $\sqrt{-25} + \sqrt{-121}$.
4. $3\sqrt{-16} - 2\sqrt{-25}$.
5. $\sqrt{-\frac{4}{25}} - \sqrt{-\frac{9}{100}}$.
6. $8\sqrt{-\frac{9}{16}} + 6\sqrt{-\frac{4}{9}}$.
7. $\sqrt{-81} + \sqrt{-36} - \sqrt{-64}$.
8. $\sqrt{-\frac{4}{9}} - \sqrt{-\frac{4}{36}} + \sqrt{-\frac{1}{4}}$.
9. $\sqrt{-x^2} - \sqrt{-4x^2} + \sqrt{-9x^2}$.
10. $\sqrt{-\frac{4}{a^2}} + \sqrt{-\frac{9}{a^2}} - \sqrt{-\frac{16}{a^2}}$.

Multiplication and Division.

420. The only special difficulty in multiplying or dividing imaginary numbers is in the sign of the product or quotient. Thus, $\sqrt{-2} \times \sqrt{-3}$ is not $\sqrt{-2 \times (-3)} = \sqrt{6}$, as in the multiplication of radicals (§ 393), but $\sqrt{6} \times (-1) = -\sqrt{6}$.

This difficulty is avoided by first reducing the imaginary numbers to the form $a\sqrt{-1}$, as below.

1. Multiply $\sqrt{-5}$ by $\sqrt{-6}$.

$$\begin{aligned} \sqrt{-5} &= \sqrt{5} \times \sqrt{-1} \\ \sqrt{-6} &= \sqrt{6} \times \sqrt{-1} \\ \hline \sqrt{-5} \times \sqrt{-6} &= \sqrt{30} \times (\sqrt{-1})^2 = \sqrt{30} \times (-1) = -\sqrt{30} \end{aligned}$$

2. Divide $\sqrt{-12}$ by $\sqrt{-3}$.

$$\begin{aligned} \sqrt{-12} &= \sqrt{12} \times \sqrt{-1} \\ \sqrt{-3} &= \sqrt{3} \times \sqrt{-1} \\ \hline \sqrt{-12} \div \sqrt{-3} &= \sqrt{4} \times 1 = \sqrt{4} \end{aligned}$$

The same result would be reached in Example 2 by proceeding as in the division of radicals (§ 397); thus, $\sqrt{-12} \div \sqrt{-3} = \sqrt{-12 \div (-3)} = \sqrt{4}$.

A reduction of terms to the form $a\sqrt{-1}$ is necessary in division only when *one* term is imaginary, and in multiplication when *both* terms are imaginary.

Multiply

3. $\sqrt{-3}$ by $\sqrt{-5}$.
4. $5\sqrt{-2}$ by $3\sqrt{-3}$.
5. $12\sqrt{-4}$ by $6\sqrt{-9}$.
6. $8\sqrt{-16}$ by $16\sqrt{-3}$.
7. $\sqrt{-\frac{4}{9}}$ by $\sqrt{-\frac{9}{16}}$.
8. $-3\sqrt{-a}$ by $5\sqrt{-b}$.
9. $\sqrt{-3}$ by $\sqrt{-4}$ by $\sqrt{-9}$.
10. $\sqrt{-6}$ by $\sqrt{-3}$ by $\sqrt{-2}$.

Divide

11. $\sqrt{-6}$ by $\sqrt{3}$.
12. $\sqrt{-2}$ by $\sqrt{-5}$.
13. $6\sqrt{3}$ by $3\sqrt{-4}$.
14. $4\sqrt{-1}$ by $-3\sqrt{-4}$.
15. $\sqrt{ab}$ by $\sqrt{-a}$.
16. $-3\sqrt{-a}$ by $2\sqrt{-4a}$.

421. The form $a \pm b\sqrt{-1}$, in which a and b are real, is called the **typical form** of imaginary numbers.

422. An imaginary number in the form $a \pm b\sqrt{-1}$ is called **complex,** and one in the form $a\sqrt{-1}$ or $\sqrt{-a}$ is called a **pure** imaginary.

423. Two complex numbers which differ only in the sign of their imaginary term are said to be **conjugate** (§ 395). Thus, $3-\sqrt{-2}$ and $3+\sqrt{-2}$ are conjugate, also $x+y\sqrt{-1}$ and $x-y\sqrt{-1}$.

424. Both the sum and the product of two conjugate complex numbers are *real.* Thus, $a+b\sqrt{-1}+a-b\sqrt{-1}=2a$, and $(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2-(-b^2)=a^2+b^2$.

It follows that the sum of two squares can be factored by introducing the imaginary unit. Thus, $a^2+b^2=(a+b\sqrt{-1})(a-b\sqrt{-1})$.

425. An imaginary number cannot equal a real number; for, if $x=\sqrt{-a}$, $x^2=-a$, which is impossible, since a positive number cannot equal a negative number.

426. If two complex imaginary numbers are equal, their real parts and their imaginary parts are respectively equal; for, if $a+b\sqrt{-1}=x+y\sqrt{-1}$, transposing, $a-x=(y-b)\sqrt{-1}$, which is impossible unless $a=x$, and $y=b$. For, if $a>$ or $<x$, then we have a real number equal to an imaginary number if $y>$ or $<b$, and to 0 if $y=b$, both of which are impossible; but, if $a=x$, then $y=b$, otherwise we have 0 equal to an imaginary number, which is impossible. Hence $a=x$, and $y=b$.

427. If the denominator of a fraction is a complex imaginary number, the fraction may be rationalized by multiplying both terms by the conjugate of its denominator, as in § 400. Thus,

$$\frac{1}{3-\sqrt{-2}}=\frac{1(3+\sqrt{-2})}{(3-\sqrt{-2})(3+\sqrt{-2})}=\frac{3+\sqrt{-2}}{9-(-2)}=\frac{3+\sqrt{-2}}{11}.$$

Simplify

17. $(3+2\sqrt{-2})(3-2\sqrt{-2})$.

18. $(\sqrt{-2}-3\sqrt{-1})(\sqrt{-2}+3\sqrt{-1})$.

19. $a^2+b\sqrt{-1}+(a+\sqrt{-b})(a-\sqrt{-b})$.

20. $(a\sqrt{-b}-c\sqrt{-b})(a\sqrt{-b}+c\sqrt{-b})$.

21. $(\sqrt{-3}+\sqrt{-2})(\sqrt{-2}-\sqrt{-4})$.

22. $(\frac{1}{2}+\frac{2}{3}\sqrt{-2})(\frac{1}{2}-\frac{2}{3}\sqrt{-2})$.

23. $\left(x-\frac{\sqrt{-3}}{2}\right)\left(x+\frac{\sqrt{-3}}{2}\right)$.

24. $(a\sqrt{-a}-b\sqrt{-b})(a\sqrt{-a}+b\sqrt{-b})$.

25. $(\sqrt{-3}+\sqrt{-2})\div(\sqrt{-3}-\sqrt{-2})$.

26. $(5-2\sqrt{-3})\div(5+2\sqrt{-3})$.

27. $\dfrac{4+3\sqrt{-2}}{4-3\sqrt{-2}}$.

28. $\dfrac{a}{a-x\sqrt{-1}}$.

29. $\dfrac{3\sqrt{-2}+2\sqrt{-3}}{3\sqrt{-2}-2\sqrt{-3}}$.

30. $\dfrac{\sqrt{-a}+\sqrt{-b}}{\sqrt{-a}-\sqrt{-b}}$.

31. $\dfrac{3}{5-2\sqrt{-2}}-\dfrac{2}{5+2\sqrt{-2}}$.

32. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}-\dfrac{a-x\sqrt{-1}}{a+x\sqrt{-1}}$.

33. $\dfrac{(x+\sqrt{-1})^2}{x-\sqrt{-1}}-\dfrac{(x-\sqrt{-1})^2}{x+\sqrt{-1}}$.

34. $2\sqrt{-1}+2\sqrt{-9}-3\sqrt{-4}$.

35. $\sqrt{-3}+\sqrt{-12}+(\sqrt{-2})(\sqrt{-2}-\sqrt{-8})$.

Expand

36. $(3-\sqrt{-2})^2$.

37. $(\sqrt{-3}-2\sqrt{-1})^2$.

38. $(5\sqrt{-1}-\sqrt{-5})^2$.

39. $(a+\sqrt{-b})^2-(a-\sqrt{-b})^2$.

40. $(x\sqrt{-x}+y\sqrt{-y})^2$.

41. $(1+\sqrt{-1})^2\div(1-\sqrt{-1})^2$.

CHAPTER XIV.

FRACTIONAL AND NEGATIVE EXPONENTS.

428. The definition of exponent already given (§ 29) applies only to exponents which are positive and integral; and the laws of exponents established have also been considered as referring to exponents which are positive and integral.

429. It remains to explain the meaning of fractional and negative exponents, as $a^{\frac{2}{3}}$ and a^{-3}, and also to show how operations are performed with numbers which contain such exponents.

FRACTIONAL EXPONENTS.

430. It is assumed that the fundamental law of exponents, $a^m \times a^n = a^{m+n}$ (§ 84), is true, whatever may be the value of m and n.

Hence, if $a^m \times a^n = a^{m+n}$, then $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$; and since $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$, $a^{\frac{1}{2}} = \sqrt{a}$; that is, $a^{\frac{1}{2}}$ denotes the square root of a.

431. Since $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$, $a^{\frac{1}{3}} = \sqrt[3]{a}$; that is, $a^{\frac{1}{3}}$ denotes the cube root of a.

Since $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \cdots$ to n factors $= a$, $a^{\frac{1}{n}} = \sqrt[n]{a}$; that is, $a^{\frac{1}{n}}$ denotes the nth root of a.

Since $a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \cdots$ to n factors $= a^{\frac{m}{n} \times n} = a^m$, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$;

and hence $a^{\frac{m}{n}}$ denotes the nth root of the mth power of a. Hence

432. *The numerator of a fractional exponent denotes the power of a number, and its denominator denotes the root of that power.*

Thus, $x^{\frac{3}{4}}$ denotes the fourth root of the cube of x.

Since $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}}$ to m factors $= a^{\frac{1}{n} \times m} = a^{\frac{m}{n}}$, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$; and hence $a^{\frac{m}{n}}$ also denotes the mth power of the nth root of a.

But this is strictly true only when the root is arithmetical or positive. For example, $\sqrt{a^4} = \pm a^2$; but $(\sqrt{a})^4$ has only one value, $+ a^2$.

433. Whatever the value of a may be, $a^0 = 1$; for $a^m \div a^m = a^{m-m} = a^0$, and $a^m \div a^m = 1$. $\therefore a^0 = 1$ (§ 123).

NEGATIVE EXPONENTS.

434. If the formula $a^m \div a^n = a^{m-n}$ is true for all values of m and n, then $a^2 \div a^4 = a^{2-4} = a^{-2}$.

But $a^2 \div a^4 = \dfrac{a^2}{a^4} = \dfrac{1}{a^2}$.

$\therefore a^{-2} = \dfrac{1}{a^2}$; that is, a^{-2} denotes the reciprocal of a^2.

Likewise $a^m \div a^{2m} = a^{m-2m} = a^{-m}$;

but $a^m \div a^{2m} = \dfrac{a^m}{a^{2m}} = \dfrac{1}{a^m}$.

$\therefore a^{-m} = \dfrac{1}{a^m}$; that is, a^{-m} denotes the reciprocal of a^m.

435. It is thus seen that *any factor may be transferred from the numerator of a fraction to its denominator, or from the denominator to the numerator, provided the sign of its exponent be changed.*

Thus, $a^{-2}b = \dfrac{b}{a^2}$; $\dfrac{a}{b^2} = ab^{-2} = \dfrac{1}{a^{-1}b^2}$; $\dfrac{a^2b^3}{x^2y^2} = \dfrac{x^{-2}y^{-2}}{a^{-2}b^{-3}}$.

Express as integers

1. $\dfrac{a^2b^3c}{a^{-2}b^{-2}}$.

2. $\dfrac{3\,a^2b^{\frac{1}{2}}}{a^{-2}b^{-\frac{1}{2}}}$.

3. $\dfrac{a^{-\frac{1}{2}}b^{-\frac{2}{3}}c}{a^{\frac{1}{2}}b^{\frac{1}{3}}}$.

4. $\dfrac{1}{x^{-\frac{1}{2}}y^{-2}z^{-\frac{1}{3}}}$.

5. $\dfrac{ab^{\frac{1}{2}}}{2^{-3}a^{-3}b^{-\frac{1}{2}}}$.

6. $\dfrac{x^{-\frac{1}{2}}y^{\frac{2}{3}}}{a^{-2}y^{-\frac{1}{3}}}$.

7. $\dfrac{2}{a^{-\frac{1}{n}}b^{-n}c^{-1}}$.

8. $\dfrac{x^ny^{\frac{n}{2}}z}{x^{-\frac{1}{n}}y^{-\frac{n}{2}}z^{-2}}$.

9. $\dfrac{a^{\frac{1}{3}}b^{\frac{1}{2}}x^2}{a^{-\frac{2}{3}}b^{-\frac{1}{2}}}$.

Express with positive exponents

10. $\dfrac{ab^2}{x^{-2}y^{-3}}$.

11. $\dfrac{a^{-2}b^{-1}}{x^{-1}y^{-2}}$.

12. $\dfrac{x^{-3}y^{-5}}{a^2b^{-2}}$.

13. $\dfrac{ab^2c^3}{x^{-3}y^{-5}}$.

14. $\dfrac{a^{\frac{1}{2}}b^2x^{-\frac{1}{3}}}{a^{\frac{1}{2}}b^{-2}x^{-\frac{2}{3}}}$.

15. $\dfrac{m^{-2}n^2x^{-4}}{m^2x^{-2}z^{-2}}$.

Express with fractional exponents

16. $\sqrt[3]{a^2b}$.

17. $\sqrt[4]{a^2x^3y}$.

18. $\sqrt[3]{-27}$.

19. $\sqrt[3]{16\,a^2x^3}$.

20. $\sqrt[3]{a^2} \times \sqrt{a^3}$.

21. $\sqrt{a^3} \div \sqrt[3]{a^2}$.

22. $\sqrt[3]{a^2b^{-\frac{1}{2}}c^{-\frac{2}{3}}}$.

23. $\sqrt[5]{a^2b^3} + \sqrt{a^2b^3}$.

24. $\sqrt[3]{a^4b^2 \div a^{-4}b^{-2}}$.

Express with radical signs and positive exponents

25. $a^{\frac{2}{3}}b^{\frac{1}{4}}$.

26. $a^{-\frac{1}{3}}b^{-2}$.

27. $5^{\frac{1}{2}}x^{-\frac{1}{2}}$.

28. $2\,a^{-\frac{1}{2}}b^2$.

29. $\dfrac{a^{-\frac{1}{3}}}{a^{-2}}$.

30. $\dfrac{3^{-2}}{4^{-\frac{1}{2}}}$.

31. $\dfrac{1}{a^{-\frac{1}{n}}}$

32. $a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{-2}$.

Find the value of

33. $8^{\frac{2}{3}}$.

34. $16^{\frac{3}{4}}$.

35. $27^{\frac{2}{3}}$.

36. $4^{-\frac{3}{2}}$.

37. $16^{-\frac{3}{2}}$.

38. $9^{\frac{3}{2}} \times 27^{-\frac{2}{3}}$.

39. $64^{\frac{1}{3}}$.

40. $100^{-\frac{1}{2}}$.

41. $1000^{\frac{2}{3}}$.

42. $36^{-\frac{3}{2}} \times 3^3$.

43. $25^{\frac{3}{2}} \div 27^{\frac{2}{3}}$.

44. $9^{-\frac{3}{2}} \times 36^{\frac{3}{2}}$.

OPERATIONS WITH FRACTIONAL AND NEGATIVE EXPONENTS.

436. The several laws of exponents may be expressed by the following formulas:

(1) $a^m \times a^n = a^{m+n}$. (4) $(a \div b)^m = a^m \div b^m$.

(2) $a^m \div a^n = a^{m-n}$. (5) $(a^m)^n = a^{mn}$.

(3) $(ab)^m = a^m b^m$. (6) $\sqrt[n]{a^{mn}} = a^m$.

437. These formulas being true for all values of m and n (§ 430), it follows that the rules previously given for operations with numbers whose exponents are positive integers, *also apply to operations with numbers which contain fractional or negative exponents.*

438. Operations with radicals may be readily performed by substituting fractional exponents for the radical signs, and then proceeding as with rational numbers (§ 402). By such substitution the operations with radical numbers may often be much simplified.

1. Multiply $5ab^{\frac{2}{3}}x^{\frac{1}{2}}y$ by $3a^{\frac{1}{3}}b^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{3}}$.

The product is found by multiplying the coefficients, and adding the exponents of the like literal factors (§ 109). The product is $15\,a^{\frac{4}{3}}b^{\frac{1}{6}}x^{\frac{9}{10}}y^{\frac{4}{3}}$.

2. Divide $a^{\frac{2}{3}}x^{\frac{5}{4}}y^{\frac{2}{3}}$ by $3\,a^{\frac{1}{3}}x^{\frac{1}{2}}y^{\frac{1}{3}}$.

$$a^{\frac{2}{3}}x^{\frac{5}{4}}y^{\frac{2}{3}} \div 3\,a^{\frac{1}{3}}x^{\frac{1}{2}}y^{\frac{1}{3}} = \frac{1}{3}\,a^{\frac{1}{3}}x^{\frac{3}{4}}y^{-\frac{1}{3}}.$$

3. Multiply $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

$$\begin{array}{r} a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b \\ a^{\frac{1}{2}} - b^{\frac{1}{2}} \\ \hline a^{\frac{3}{2}} + ab^{\frac{1}{2}} + a^{\frac{1}{2}}b \\ \quad - ab^{\frac{1}{2}} - a^{\frac{1}{2}}b - b^{\frac{3}{2}} \\ \hline a^{\frac{3}{2}} \qquad\qquad - b^{\frac{3}{2}} \end{array}$$

4. Divide

$$x^{\frac{1}{2}} - 3x^{\frac{1}{3}}y^{-\frac{1}{6}} + 3x^{\frac{1}{6}}y^{-\frac{1}{3}} - y^{-\frac{1}{2}} \text{ by } x^{\frac{1}{3}} - 2x^{\frac{1}{6}}y^{-\frac{1}{6}} + y^{-\frac{1}{3}}.$$

$$\begin{array}{l|l} x^{\frac{1}{2}} - 3x^{\frac{1}{3}}y^{-\frac{1}{6}} + 3x^{\frac{1}{6}}y^{-\frac{1}{3}} - y^{-\frac{1}{2}} & x^{\frac{1}{3}} - 2x^{\frac{1}{6}}y^{-\frac{1}{6}} + y^{-\frac{1}{3}} \\ \underline{x^{\frac{1}{2}} - 2x^{\frac{1}{3}}y^{-\frac{1}{6}} + x^{\frac{1}{6}}y^{-\frac{1}{3}}} & x^{\frac{1}{6}} - y^{-\frac{1}{6}} \\ -x^{\frac{1}{3}}y^{-\frac{1}{6}} + 2x^{\frac{1}{6}}y^{-\frac{1}{3}} - y^{-\frac{1}{2}} & \\ \underline{-x^{\frac{1}{3}}y^{-\frac{1}{6}} + 2x^{\frac{1}{6}}y^{-\frac{1}{3}} - y^{-\frac{1}{2}}} & \end{array}$$

Multiply

5. $a^{\frac{2}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{2}{3}} - b^{\frac{2}{3}}$.

6. $a^{\frac{5}{6}} + b^{-\frac{2}{3}}$ by $a^{\frac{5}{6}} - b^{-\frac{2}{3}}$.

7. $x + x^{-1}$ by $x + x^{-1}$.

8. $x^{\frac{1}{2}} + y^{-\frac{1}{2}}$ by $x^{-\frac{1}{2}} + y^{\frac{1}{2}}$.

9. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

10. $x^{-\frac{2}{3}} + x^{-\frac{1}{3}} + 1$ by $x^{-\frac{1}{3}} - 1$.

11. $x + x^{\frac{1}{2}} + 2$ by $x + x^{\frac{1}{2}} - 2$.

12. $x^4 + x^2 + 1$ by $x^{-4} - x^{-2} + 1$.

Divide

13. $a - b$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

14. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

15. $27x^{-2} + 8y^{-3}$ by $3x^{-\frac{2}{3}} + 2y^{-1}$.

16. $x^{\frac{4}{3}} - y^{\frac{4}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

17. $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

18. $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{4}}$.

Extract the square root of

19. $4a - 4a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.

20. $x^{-1} + 2x^{-\frac{1}{2}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

21. $x^{\frac{5}{3}} - 4x^{\frac{4}{3}} + 2x + 4x^{\frac{2}{3}} + x^{\frac{1}{3}}$.

22. $x^{-4} - x^{-3}y^{-1} + \frac{9x^{-2}y^{-2}}{4} - x^{-1}y^{-3} + y^{-4}$.

Simplify 23. $(a^3)^{-2} \times (a^{-3})^{\frac{2}{3}}$.

24. $\left(3^n - 2 + \frac{1}{3^n}\right)^{\frac{1}{2}}$.

25. $\sqrt{(x^{\frac{1}{4}}y^{\frac{1}{2}})^8 x^{-2}y^{-3}}$.

26. $\sqrt{(a^{\frac{1}{3}}b^{\frac{1}{3}}c)^{-12}a^2b^{-\frac{5}{2}}}$.

27. $(a^{-\frac{a}{b}}y^{-1})^{-b} \times \left(\frac{x^{a^2-b^2}}{y^{ab-b^2}}\right)^{\frac{1}{a-b}}$.

28. $\{(a^m)^{m-\frac{1}{m}}\}^{\frac{1}{m+1}} \times \frac{a^{-(m-1)}}{b^{-m}}$.

CHAPTER XV.

QUADRATIC EQUATIONS.

439. An equation which, when cleared of fractions, contains the square of the unknown number, but no higher power, is *an equation of the second degree.* An equation of the second degree is called a **quadratic equation.** Thus, $4x^2 = 16$, and $3x^2 - 4x = 15$, are quadratic equations.

440. A quadratic equation may be reduced to the general form $ax^2 + bx + c = 0$, in which a, b, and c are known numbers. The known numbers, a, b, are called the *coefficients* of the equation; and the term c, the *constant* term.

The term c may also be made a coefficient by writing the equation in the form of $ax^2 + bx + cx^0 = 0$.

441. When the coefficient b is zero, the equation becomes $ax^2 + c = 0$, and is said to be *incomplete*, because the first power of x is wanting. When no one of the coefficients is zero, the equation is said to be *complete.* Hence

An **incomplete quadratic equation** contains only the square of the unknown number.

A **complete quadratic equation** contains both the square and the first power of the unknown number.

An incomplete quadratic equation is also called a *pure quadratic;* and a complete quadratic equation, an *affected quadratic.*

442. The *root* of an equation is the value of the unknown number in it (§ 138). When a root is substituted for the unknown number, *it satisfies the equation;* that is, reduces it to an *identity.*

INCOMPLETE QUADRATICS.

443. An incomplete quadratic equation is reduced, if necessary, to the general form $ax^2 = c$ by the same transformations as those employed in the solution of simple equations.

444. An equation of the form $ax^2 = c$ is solved *by dividing both terms by* a, *and then extracting the square root of both members.*

Thus, $ax^2 = c$;

dividing by a, $x^2 = \frac{c}{a}$;

extracting square root, $x = \pm\sqrt{\frac{c}{a}}$.

445. The substitution of $+\sqrt{\frac{c}{a}}$ or $-\sqrt{\frac{c}{a}}$ for x will satisfy the equation (§ 442); and hence *an incomplete quadratic equation has two roots, numerically equal, but having opposite signs.*

446. A root which can be exactly found, as $\sqrt{9}$, is a *rational* root.

A root which can only be found approximately, as $\sqrt{5}$, is an *irrational root*, or *surd* (§ 371).

A root which cannot be found either exactly or approximately, as $\sqrt{-9}$, is an *imaginary* root (§ 413).

1. Solve the equation $\frac{x+3}{x-3} + \frac{x-3}{x+3} = \frac{29}{10}$.

Clearing of fractions, $10(x+3)^2 + 10(x-3)^2 = 29(x^2-9)$;

expanding, $10x^2 + 60x + 90 + 10x^2 - 60x + 90 = 29x^2 - 261$;

transposing and simplifying, $x^2 = 49$;

extracting square root, $x = \pm 7$.

2. Solve the equation $\frac{5x^2-5}{2}=2x^2-7$.

Clearing of fractions, transposing, etc., whence

$$5x^2-5=4x^2-14;$$
$$x^2=-9;$$
$$x=\pm\sqrt{-9}=\pm 3\sqrt{-1}.$$

Since the square root of a negative number is imaginary (§ 413), the value of x can only be indicated.

3. Solve the equation $x-\sqrt{x^2-3}=\frac{2}{\sqrt{x^2-3}}$.

Clearing of fractions, transposing and uniting, squaring each member, transposing and simplifying, whence

$$x\sqrt{x^2-3}-(x^2-3)=2;$$
$$x\sqrt{x^2-3}=x^2-1;$$
$$x^4-3x^2=x^4-2x^2+1;$$
$$x^2=-1;$$
$$x=\pm\sqrt{-1}.$$

Solve the equations

4. $7x^2=567$.

5. $11x^2-9=35$.

6. $5x^2+18=7x^2$.

7. $\frac{3x^2}{4}-5=7$.

8. $\frac{x^2+5}{4}=\frac{2x^2-11}{2}$.

9. $\frac{x+3}{4}=\frac{4}{x-3}$.

10. $\frac{x^2-9}{4}=10$.

11. $x+\frac{1}{x}=2x$.

12. $\frac{x-5}{3x}=\frac{2x}{x+5}$.

13. $\frac{x^2-4}{3}=\frac{x^2+11}{4}$.

14. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=3$.

15. $\sqrt{x^2+2}=\frac{8}{\sqrt{x^2-2}}$.

16. $\frac{1}{1+x}+\frac{1}{1-x}=\frac{1}{5}$.

17. $\frac{2}{\sqrt{x^2-1}}=\frac{1}{\sqrt{x^2-7}}$.

18. $\frac{1}{2x^2}+2=\frac{90}{4x^2}$.

19. $\frac{x-2}{2x}=\frac{x}{x+2}$.

20. $\frac{(3x^2-9)^2}{3}=3x^4-9$.

21. $\frac{x+5}{x-5}+\frac{x-5}{x+5}=-\frac{15}{4}$.

22. $\dfrac{\sqrt{1+x^2}-x}{x}=\dfrac{1}{4}$.

23. $\dfrac{m}{x^2-n}=\dfrac{n}{x^2-m}$.

24. $\sqrt{a+x}=\dfrac{a}{\sqrt{a-x}}$.

25. $\dfrac{x}{a}-\dfrac{a}{x}=ax$.

26. $\dfrac{x^2+50}{5}=x^2-\dfrac{x^2-10}{3}$.

27. $\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}=\dfrac{b}{c}$.

28. $\dfrac{a}{x+b}-\dfrac{a}{x-b}=c$.

29. $\dfrac{1}{1-\sqrt{1-x^2}}-\dfrac{\sqrt{3}}{x^2}=\dfrac{1}{1+\sqrt{1-x^2}}$.

30. $\sqrt{\dfrac{x^2-16}{x-3}}+\sqrt{x+3}=\dfrac{7}{\sqrt{x-3}}$.

31. Two numbers are to each other as 3 to 5, and the difference of their squares is 256. What are the numbers?

SUGGESTION. Let $3x$ and $5x$ be the numbers.

32. Two numbers are to each other as 2 to 7, and their product is 126. What are the numbers?

33. A rectangular field contains 6 acres, and its sides are to each other as 3 to 4. What is the length of each side?

34. The length of a rectangular field is $2\frac{1}{2}$ times the width, and the field contains 9 acres. What is the length of each side?

35. The sum of two numbers is 16, and their product is 60. What are the numbers?

SUGGESTION. Let $8+x$ and $8-x$ be the numbers.

36. A father's age is to his son's age as 5 to 2, and the product of their ages is 640. What are their ages?

37. Two numbers are to each other as a to b, and their product is c^2. What are the numbers?

38. A son's age is to his father's age as a to b, and the product of their ages is m. What are their ages?

COMPLETE QUADRATICS.

447. There are, as will be seen, several methods of solving complete quadratic equations; and it is advisable for pupils, especially those who expect to enter higher institutions, to acquire a knowledge of those in more common use, though in practice it may not be best to use more than two of them.

Method of Solution by Factoring.

448. A product is zero if any one of its factors is zero (§ 22); and, in order that *any product may be zero, at least one of its factors must be zero.*

For example, if $(x-2)(x+3)=0$, then $x-2=0$, or $x+3=0$. But if $x-2=0$, $x=2$; and if $x+3=0$, $x=-3$; and, since either value of x thus found will satisfy the equation $(x-2)(x+3)=0$, its roots are 2 and -3.

449. It is thus seen that any quadratic equation whose first member is a product, and whose second member is zero, may be solved *by equating each of its two factors to zero, and solving the resulting equations.*

It is here assumed that the factors are finite.

1. What are the roots of the equation $(x+3)(x-1)=0$?

Equating the factors to zero, $x+3=0$, and $x-1=0$;

whence $x=-3$, and $x=1$.

Hence the roots are -3 and 1.

Find the roots of

2. $(x-7)(x-2)=0$.
3. $(x+1)(x-5)=0$.
4. $(x-6)(x-4)=0$.
5. $(x-\frac{1}{2})(x+2)=0$.
6. $(x+\frac{1}{3})(x-\frac{1}{2})=0$.
7. $(x-b)(x+c)=0$.
8. $(x-a)(x-\sqrt{b})=0$.
9. $(x+\sqrt{a})(x-\sqrt{a})=0$.

450. A quadratic equation of the form $x^2 + bx + c = 0$ may be readily factored by § 202, and its roots thus found, provided c is the product of two rational factors whose algebraic sum is b.

For example, let $x^2 + 3x - 10 = 0$.

Factoring by § 202, $(x + 5)(x - 2) = 0$;

equating factors to zero, $x + 5 = 0$, and $x - 2 = 0$;

whence $x = -5$, and $x = 2$.

Hence the roots of the equation are -5 and 2.

10. Solve the equation $x^2 - 7x = -12$.

Transposing -12, $x^2 - 7x + 12 = 0$;

factoring, $(x - 4)(x - 3) = 0$;

equating to zero, $x - 4 = 0$, and $x - 3 = 0$;

whence $x = 4$, and $x = 3$.

Hence the roots are 4 and 3.

451. A quadratic equation of the form $ax^2 + bx + c = 0$ may be factored by § 205 or by the general formula, § 208, and its roots be thus found.

11. Solve the equation $3x^2 + 7x = 6$.

Transposing 6, $3x^2 + 7x - 6 = 0$;

factoring by § 205, $(3x - 2)(x + 3) = 0$;

equating to zero, $3x - 2 = 0$, and $x + 3 = 0$;

whence $x = \frac{2}{3}$, and $x = -3$.

Hence the roots are $\frac{2}{3}$ and -3.

Solve by factoring the equations

12. $x^2 - x = 12$.
13. $x^2 - 4x = 45$.
14. $x^2 - 10x = -21$.
15. $x^2 - 12x = -32$.
16. $x^2 + 3x = 28$.
17. $x^2 + x = 56$.
18. $x^2 - 10x = 56$.
19. $x^2 + 15x = -26$.
20. $x^2 - 15x = 154$.
21. $2x^2 + 6x = 20$.
22. $3x^2 - 7x = 40$.
23. $5x^2 + 27x = 18$.

24. $6x^2 - 7x = 20$.

25. $4x^2 - 11x = -7$.

26. $4x^2 - 23x = -15$.

27. $x^2 + ax + bx = -ab$.

28. $x^2 + 2a^2 - 3ax = 0$.

29. $x^2 - \left(a - \frac{b}{2}\right)x = \frac{1}{2}ab$.

30. $x^2 + (\sqrt{a} + \sqrt{b})x = -\sqrt{ab}$.

452. *Any* complete quadratic equation may be solved by factoring, but the method has no special advantage when the factors cannot be determined *by inspection.* When the factors can be thus readily determined, no other method of solution need be used.

453. When a quadratic equation can be readily factored by inspection, its roots may be written at once without equating the factors to zero.

Thus, the roots of $x^2 - 2x - 24 = 0$ are seen to be -4 and 6.

Write, without equating factors, the roots of

31. $x^2 + 6x + 9 = 0$.

32. $x^2 - 5x + 6 = 0$.

33. $x^2 + 3x - 10 = 0$.

34. $x^2 - 7x - 60 = 0$.

35. $2x^2 + 10x + 8 = 0$.

36. $3x^2 - 12x - 15 = 0$.

454. Conversely, if the two roots of a complete quadratic are given, the equation can be formed *by finding the product of the two binomial factors whose first terms are* x, *and whose second terms are respectively the two given roots with signs changed, and then writing the product thus found equal to zero.*

Thus, if -2 and 3 are the two roots of a complete quadratic, the equation is $(x + 2)(x - 3) = 0$, or $x^2 - x - 6 = 0$.

It can be shown that $x^2 - x - 6 = 0$ is the *only* equation whose roots are -2 and 3 with no other roots, but the proof is too difficult for insertion here. $3x^2 - 3x - 18 = 0$ is reducible to $x^2 - x - 6 = 0$, and hence is no exception.

37. Form the quadratic equation whose roots are 3 and -5.

$x = 3$, and $x = -5$.

Transposing 3 and -5, $x - 3 = 0$, or $x + 5 = 0$;

whence $(x - 3)(x + 5) = 0$;

$\therefore x^2 + 2x - 15 = 0$.

38. Form the quadratic whose roots are $\frac{2}{3}$ and $-\frac{1}{2}$.

$x = \frac{2}{3}$, and $x = -\frac{1}{2}$.

Transposing, $x - \frac{2}{3} = 0$, and $x + \frac{1}{2} = 0$;

whence $(x - \frac{2}{3})(x + \frac{1}{2}) = 0$;

$\therefore\ x^2 - \frac{1}{6}x - \frac{1}{3} = 0$;

clearing of fractions, $6x^2 - x - 2 = 0$.

455. The required equation may be written at once by changing the signs of the given roots, and then making their algebraic *sum* the coefficient of x, and their *product* the third term.

Thus, if the two given roots are a and $-\frac{a}{b}$, the equation is $x^2 + \left(-a + \frac{a}{b}\right)x - \frac{a^2}{b} = 0$.

Form the quadratic equation whose roots are

39. 1 and -5.

40. -3 and -4.

41. -2 and $\frac{1}{2}$.

42. $\frac{1}{3}$ and $-\frac{1}{4}$.

43. $-\frac{2}{3}$ and $-\frac{1}{2}$.

44. -4 and 0.

45. $-\sqrt{5}$ and $\sqrt{5}$.

46. $\sqrt{3}$ and $-\sqrt{3}$.

47. $2 - \sqrt{3}$ and $2 + \sqrt{3}$.

48. $\frac{a}{b}$ and $-a$.

Solution by Completing the Square.

456. Any complete quadratic equation may be reduced to the form $x^2 \pm bx = c$ by dividing, if necessary, both members by the coefficient of x^2.

457. If the square of $\frac{1}{2}$ of b, or $\left(\frac{b}{2}\right)^2$, be added to both members of $x^2 + bx = c$ (1), called **completing the square,** the equation becomes

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \frac{b^2}{4}. \qquad (2)$$

The first member of (2) is now a *perfect square,* the extremes being each a perfect square, and the middle term twice the product of their square roots (§ 186).

Extracting the square root of each member of (2), we have

$$x+\frac{b}{2}=\pm\sqrt{c+\frac{b^2}{4}};$$

and transposing $+\frac{b}{2}$, $\qquad x=-\frac{b}{2}\pm\sqrt{c+\frac{b^2}{4}}.$

It is thus seen that the equation has two roots,

$$-\frac{b}{2}+\sqrt{c+\frac{b^2}{4}} \text{ and } -\frac{b}{2}-\sqrt{c+\frac{b^2}{4}}.$$

It follows from § 454 that the first member of the equation $x^2+bx-c=0$ is the product of $x+\left(\frac{b}{2}+\sqrt{c+\frac{b^2}{4}}\right)$ and $x+\left(\frac{b}{2}-\sqrt{c+\frac{b^2}{4}}\right)$.

1. Solve the quadratic $3x^2-12x=15$.

Dividing by 3, $\quad x^2-4x=5;$
completing the square, $\quad x^2-4x+(2)^2=5+4=9;$
extracting the square root, $\quad x-2=\pm 3;$
whence $\quad x=2\pm 3=5 \text{ or } -1.$

2. Solve the quadratic $x^2+11x=33\frac{3}{4}$.

Completing the square, $x^2+11x+(\frac{11}{2})^2=1\frac{3}{4}\frac{5}{4}+1\frac{2}{4}\frac{1}{4}=2\frac{5}{4}\frac{6}{4};$
extracting square root, $\quad x+\frac{11}{2}=\pm\frac{16}{2};$
whence $\quad x=-\frac{11}{2}\pm\frac{16}{2}=2\frac{1}{2} \text{ or } -13\frac{1}{2}.$

3. Solve the quadratic $\dfrac{3x-7}{x}-\dfrac{4x+2}{5}=2x-3.$

Clearing of fractions, etc., $\quad -14x^2+28x=35;$
dividing by -14, $\quad x^2-2x=-\frac{5}{2};$
completing the square, $\quad x^2-2x+1=-\frac{5}{2}+1=-\frac{3}{2};$
extracting the square root, $\quad x-1=\pm\sqrt{-\frac{3}{2}};$
$\therefore\ x=1\pm\sqrt{-\frac{3}{2}}=1\pm\frac{1}{2}\sqrt{-6}.$

The two roots are imaginary, as they will always be when $c+\frac{b^2}{4}$ is *negative* (§ 413).

Solve by completing the square

4. $x^2 - 8x = -15$.
5. $x^2 + 12x = -20$.
6. $x^2 + 4x = 21$.
7. $x^2 - 12x = 45$.
8. $x^2 - 20x + 19 = 0$.
9. $x^2 + 18x - 88 = 0$.
10. $2x^2 - 10x + 6 = 11x - x^2 - 30$.
11. $3x^2 - 11x = 20$.
12. $5x^2 - x + 19 = 3x^2 + 15x - 15$.
13. $7x^2 - 12x = 580$.
14. $(3x + 10)(3x + 7) = 2(2 - 2x - x^2)$.
15. $2x^2 + 94x + 420 = 0$.
16. $3x^2 - 52x + 118 = (5 - 2x)(3x + 2)$.
17. $2(x - 3)(x - 4) = x^2 - 25$.
18. $\dfrac{x + 32}{x^2 - 2x - 20} = \dfrac{39}{2x + 1}$.
19. $\dfrac{5x + 3}{8x - 4} = \dfrac{7x + 2}{11x - 3}$.
20. $\dfrac{1}{2x^2} + 7 = \dfrac{9}{x}$.
21. $\dfrac{7x^2}{4} + \dfrac{2x^2 - 15}{4} = \dfrac{4x^2 + 5}{2}$.
22. $2(x - 2)(x - 3) = (x - 4)(x - 3) + 10$.
23. $35 - \frac{1}{5}(x^2 + 50) = x^2 - \frac{1}{3}(x^2 - 10)$.

458. To solve a complete quadratic by completing the square,

Reduce the equation to the form $x^2 + bx = c$.

Add the square of one half the coefficient of x *to both members, thus completing the square of the first member.*

Extract the square root of each member of the resulting equation, and then find the two values of x.

The problem of solving a complete quadratic equation is thus reduced to that of solving two simple equations.

Other Methods of Completing the Square.

Note. The four sections following (§§ 459–462) may be omitted by beginners.

459. Instead of dividing both members of the equation by the coefficient of x^2, it is sometimes more convenient *to multiply both members, if necessary, by such a number as will make the coefficient of* x^2 *a perfect square*, and then complete the square *by adding to each member the square of the quotient obtained by dividing the coefficient of* x *by twice the square root of the coefficient of* x^2.

Take, for example, the equation $3x^2 - 10x = -3$.

Multiplying both members by 3, we have,

$$9x^2 - 30x = -9;$$

completing square by adding $\left(\frac{30}{3 \times 2}\right)^2 = 5^2$,

$$9x^2 - 30x + 5^2 = -9 + 25 = 16;$$

extracting the square root, $3x - 5 = \pm 4;$

transposing and uniting, $3x = 5 \pm 4 = 9 \text{ or } 1;$

$$\therefore x = 3 \text{ or } \tfrac{1}{3}.$$

When the coefficient of x^2 is already a perfect square, the square may be at once completed as above.

1. Solve the equation $18x^2 - 15x = 42$.

Multiplying both members by 2, $36x^2 - 30x = 84;$

completing the square by adding $(\frac{30}{12})^2 = (\frac{5}{2})^2$,

$$36x^2 - 30x + (\tfrac{5}{2})^2 = 84 + (\tfrac{5}{2})^2 = \tfrac{361}{4};$$

extracting the square root, $6x - \frac{5}{2} = \pm \frac{19}{2};$

transposing and uniting, $6x = \frac{5}{2} \pm \frac{19}{2} = 12 \text{ or } -7;$

$$\therefore x = 2 \text{ or } -1\tfrac{1}{6}.$$

Solve as above the equations

2. $8x^2 - 12x = 80.$
3. $3x^2 - 8x + 4 = 0.$
4. $10x^2 + 2 = -12x.$
5. $7x^2 - 28x = -21.$

6. $9x^2 + 9x + 2 = 0$.
7. $27x^2 - 30x = 48$.
8. $5x^2 - 44x = 9$.
9. $12x^2 = 24x + 420$.
10. $32x^2 - 60x = 272$.
11. $25x = 6x^2 + 21$.
12. $50x^2 - 27 = 15x$.
13. $18x^2 - 27x - 26 = 0$.

460. The foregoing methods of completing the square involve fractions when twice the square root of the coefficient of x^2 is not a factor of the coefficient of x; and these fractions sometimes necessitate much work, as will be seen by solving by either method the equation $29x^2 - 31x = 54$.

461. The first member of any complete quadratic equation may be made a complete square without fractions.

For, take an equation of the general form

$$ax^2 + bx = c. \tag{1}$$

Multiplying both members by $4a$ (4 times the coefficient of x^2), we have

$$4a^2x^2 + 4abx = 4ac. \tag{2}$$

Since $4ab \div 2\sqrt{4a^2} = b$, the square of the first member of (2) is completed by adding b^2 (the square of the coefficient of x in (1), the given equation) to both members; thus,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2. \tag{3}$$

Extracting the square root, we have

$$2ax + b = \pm\sqrt{4ac + b^2}. \tag{4}$$

Solving the simple equation (4),

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

462. Hence any complete quadratic equation may be solved without fractions as follows:

Multiply each member by 4 times the coefficient of x^2.

Add to each member of the resulting equation the square of the coefficient of x *in the given equation.*

Extract the square root of both members of this equation, and solve the resulting simple equation.

This method was first used by a Hindoo mathematician, and for this reason it is known as the *Hindoo method.*

14. Solve the equation $13x^2 - 15x = 22$.

Multiplying by 4·13, $4 \cdot 13^2 x^2 - 4 \cdot 13 \cdot 15 x = 4 \cdot 13 \cdot 22 = 1144$;

completing the square by adding 15^2, $4 \cdot 13^2 x^2 - 4 \cdot 13 \cdot 15 x + 15^2 = 1144 + 15^2 = 1369$;

extracting square root, $26x - 15 = \pm 37$;

transposing, etc., $26x = 52$ or -22;

$\therefore \quad x = 2$ or $-\frac{11}{13}$.

When the coefficient of x in the given equation is an even number, the square of the first member may be completed without fractions *by multiplying both members by the coefficient of* x², *and then adding to each member the square of half the coefficient of* x *in the given equation.*

Solve the quadratic equations

15. $3x^2 + 2x = 2\frac{2}{3}$.

16. $13x^2 - 24x = 205$.

17. $15x^2 - 207 = 24x$.

18. $-10x^2 + 23x = 12$.

19. $21x^2 + 1 = -10x$.

20. $15x^2 - 8x + 1 = 0$.

21. $20x^2 - 54x = 104$.

22. $11x^2 - 24 = 10x$.

23. $3x^2 + 2x = 56$.

24. $5x^2 - 8 = 6x$.

Method of Solution by Formula.

463. The solution of a complete quadratic equation of the general form $ax^2 + bx = c$ gives as the two roots

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

Instead of working out the solution of every equation from the beginning, completing the square, etc., we may write out the roots at once *by substituting for* a, b, *and* c *in the above general formula their values in the particular equation,* as shown below.

1. Solve the equation $x^2 - 11x = -24$.

In this case $a = 1$, $b = -11$, and $c = -24$.

Substituting these values of a, b, and c, in the formula, we have

$$x = \frac{11 \pm \sqrt{11^2 + 4(-24)}}{2} = \frac{11 \pm \sqrt{121 - 96}}{2} = \frac{11 \pm \sqrt{25}}{2};$$

$$\therefore\ x = \frac{11 \pm 5}{2} = 8 \text{ or } 3.$$

In substituting, special attention must be given to the *signs* of the coefficients.

2. Solve the equation $17x^2 + 8x = 21$.

$$a = 17,\ b = 8, \text{ and } c = 21.$$

Substituting these values in the formula,

$$x = \frac{-8 \pm \sqrt{64 + 1428}}{34} = \frac{-8 \pm \sqrt{1492}}{34};$$

$$\therefore\ x = \frac{-8 \pm 38.6264+}{34} = .9007 \text{ or } -1.3713.$$

Since $\sqrt{1492}$ is an irrational number, the two roots of x are surds.

464. The correctness of the roots obtained may be verified by substituting them for x in the given equation; but it is usually better to verify by the aid of the principle stated in § 466, that the sum of the two roots in the general formula $= \frac{-b}{a}$, and their product $= \frac{-c}{a}$.

Thus, in Example 2, above, $\frac{-8+\sqrt{1492}}{34} + \frac{-8-\sqrt{1492}}{34} = \frac{-8}{17}$,

and $\frac{-8+\sqrt{1492}}{34} \times \frac{-8-\sqrt{1492}}{34} = \frac{-21}{17}$.

In using this method, a glance at the coefficients of the equation will usually show the correctness of the results.

Solve by formula and verify the roots of

3. $x^2 - 24x = 481$.

4. $x^2 - 41x - 348 = 0$.

5. $x^2 - 24x = -119$.

6. $3x^2 - 11x = 4$.

7. $2x^2 + 38x = 364$.
8. $4x^2 - 17x = 42$.
9. $5x^2 + 21x = 62$.
10. $7x^2 + 34x + 24 = 0$.
11. $3x^2 - 11x - 20 = 0$.
12. $-2x^2 + 4x - 5 = 0$.
13. $mx^2 + nx - 4 = 0$.
14. $m^2x^2 + mx - 1 = 0$.

465. Assuming that the coefficients in the general equation $ax^2 + bx = c$ are finite, and not zero, it can be shown, by methods that properly belong to higher algebra, that it has *two roots, and only two.*

466. The roots of a quadratic equation have the following general properties:

I. Adding the two roots as expressed in the formula $\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$, it is found that their *sum* is $\frac{-b}{a}$; and, multiplying them, it is found that their *product* is $\frac{-c}{a}$.

II. If $b^2 + 4ac$ be positive, $\sqrt{b^2 + 4ac}$ is *real,* and the roots of the equation are *real* and *unequal.*

III. If $b^2 + 4ac$ be a perfect square, the two roots are *rational;* if $b^2 + 4ac$ be not a perfect square, the two roots are *irrational.*

IV. If $b^2 + 4ac$ is negative, both roots are *imaginary.*

V. If $b^2 + 4ac = 0$, both roots reduce to $\frac{-b}{2a}$, and are thus *equal;* and it is then said that the equation has *two equal roots.*

LITERAL QUADRATICS.

1. Solve the equation $(a - x)(a + x) = a(x - a)$.

Multiplying, transposing, etc., $x^2 + ax - 2a^2 = 0$;
factoring, $(x - a)(x + 2a) = 0$;
equating factors, $x - a = 0$, and $x + 2a = 0$;
$\therefore x = a$ or $-2a$.

2. Solve the equation $x^2 + mx - nx = mn$.

Completing the square, $x^2+(m-n)x+\left(\frac{m-n}{2}\right)^2 = mn + \frac{m^2-2mn+n^2}{4}$

$$= \frac{m^2+2mn+n^2}{4};$$

extracting the square root, $x + \frac{m-n}{2} = \pm\frac{m+n}{2}$;

transposing, $x = -\frac{m-n}{2} \pm \frac{m+n}{2}$.

$\therefore\ x = n$ or $-m$.

3. Solve by formula the equation $ax^2 - (a^2 - b)x = ab$.

$$x = \frac{-[-(a^2-b)] \pm \sqrt{(a^2-b)^2 + 4a^2b}}{2a} = \frac{(a^2-b) \pm (a^2+b)}{2a}.$$

$$\therefore\ x = \frac{2a^2}{2a} = a, \text{ or } \frac{-2b}{2a} = -\frac{b}{a}.$$

Solve the following equations by any one of the above methods:

4. $x^2 - ax - x = -a$.

5. $x^2 - \frac{a^2+b^2}{ab}x = -1$.

6. $x + \frac{a^2b^2}{x} = a^2 + b^2$.

7. $ax^2 - 2ax + x = 2$.

8. $\frac{a+x}{a} + \frac{a}{a-x} - 1 = 0$.

9. $x^2 - (n+1)x = -n$.

10. $\frac{x^2}{x+1} = \frac{a^2}{a+1}$.

11. $x + \frac{ab}{x} = a + b$.

12. $x - 4ab = \frac{(a+b)(a-b)}{x}$.

13. $\frac{a}{a+x} - \frac{a}{a-x} = \frac{ax}{a-x}$.

14. $a^2(x^2 + a^2) = 2a^3x + 1$.

15. $x^2 - 2ax + bx - 2ab = 0$.

16. $x^2 - (a - b - c)x = ac + ab$.

17. $x^2 - (2c + 2d)x = 4cd - 3c^2 - d^2$.

18. $(a+b)x^2 - 2(a-b)x = \frac{4ab}{a+b}$.

19. $\frac{x+1}{\sqrt{x}} = \frac{m+1}{\sqrt{m}}$.

Miscellaneous Exercises.

467. Solve the following quadratic equations, each by the method that may seem best adapted to its solution. Verify the results by seeing whether the sum of the two roots is $\frac{-b}{a}$, and their product $\frac{c}{a}$ (§ 464); or by direct substitution.

1. $x^2 - 21x + 68 = 0.$
2. $x^2 - x - 182 = 0.$
3. $2x^2 - 3x - 5 = 0.$
4. $3x^2 - 8x = -4.$
5. $9x^2 - 12x + 4 = 0.$
6. $5x^2 + 7x = 160.$
7. $12x^2 - 26x = -13.$
8. $x^2 + 2a^2 = 3ax.$
9. $2x^2 + 3x = 2.$
10. $12x^2 - x - 1 = 0.$
11. $x^2 + \frac{7x}{6} + \frac{1}{3} = 0.$
12. $x^2 + 4x = 0.$
13. $x^2 + \frac{x}{6} - \frac{1}{3} = 0.$
14. $x^2 - \frac{x}{2} - 5 = 0.$
15. $x + \frac{1}{a} = a + \frac{1}{x}.$
16. $\frac{a}{a+x} + \frac{a}{a-x} = 4.$
17. $bx^2 - (a + ab)x = -a^2.$
18. $x^2 - 2ax = b - a^2.$
19. $\frac{a^2}{x-b} + \frac{b^2}{x-a} = a + b.$
20. $\frac{x}{a} + \frac{a}{x} = \frac{b}{a} + \frac{a}{b}.$
21. $x^2 + ax - bx - ab = 0.$
22. $x^2 - cx - \frac{2cx}{a} + \frac{2c^2}{a} = 0.$
23. $x^2 - 2(a-b)x + b^2 = 2ab.$
24. $\frac{1}{x-a} - \frac{2}{x+a} = 1.$
25. $\frac{x}{x-2} + \frac{4}{x+1} = 3.$
26. $\frac{1}{b-x} - \frac{1}{b+x} = \frac{3+x^2}{b^2-x^2}.$
27. $x^2 - 2ax + 12x = 24a.$
28. $\frac{x}{a} + \frac{2a}{x} = \frac{3}{a}.$
29. $x + \frac{1}{x} = \frac{13}{6}.$
30. $5x(x-3) = \frac{21}{4} - 12x.$
31. $\frac{3x+4}{5} - \frac{30-2x}{x-6} = \frac{7x-14}{10}.$
32. $\frac{10}{x} - \frac{14-2x}{x^2} = \frac{5}{2}.$

33. $\frac{x+4}{3}-\frac{4x+7}{9}=\frac{10-2x}{x-3}$.

34. $\frac{x-4}{5}+\frac{5}{x-4}=x+11$.

35. $10(x+2)=\frac{30x}{x-2}$.

36. $\frac{x-1}{x-2}+\frac{x-2}{x-3}=\frac{7}{2}$.

37. $3(x+4)^2+2(x-17)^2=274$.

38. $x^2+4nx=4x^2+1$.

39. $x^2-ax=a-x$.

40. $x^2-7ax=78a^2$.

41. $\frac{x}{x+a}+\frac{x+a}{x}=\frac{5}{2}$.

42. $x-\frac{1}{x}=n$.

43. $ax+\frac{b}{x}=c$.

44. $(m+n)x^2-\frac{4mn}{m-n}x=m+n$.

45. $x^2-n^2=m(2x-m)$.

46. $c^2x^2-acx=\frac{1}{4}(b-a^2)$.

47. $2x-5\sqrt{x}=3$.

48. $2x+1=\sqrt{6x+3}$.

49. $x+2\sqrt{3}x+1=0$.

50. $\sqrt{a-x}+\sqrt{b-x}=\frac{b}{\sqrt{b-x}}$.

51. $\sqrt{x}+\sqrt{5x+1}=2$.

52. $x+2\sqrt{x-5}=5$.

53. $\sqrt{x+1}+\frac{1}{\sqrt{x+1}}=2$.

54. $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{x+1}}$.

55. $\frac{x+1}{\sqrt{x}}=\frac{c+1}{\sqrt{c}}$.

56. $\sqrt{x-a}=\frac{a^2}{\sqrt{x+2a}}$.

57. $2\sqrt{x^2-64}=2x-8$.

58. $\sqrt{x+5}-\sqrt{x}=1$.

59. $\sqrt{2x+4}=\sqrt{\frac{x}{2}+6}+1$.

60. $\sqrt{2x+1}=\frac{x+2}{\sqrt{2x+1}}+1$.

61. $(5x-2)^2-(3x+2)^2=(x-3)^2-1$.

62. $\frac{11(x+3)}{x+4}=\frac{4(x-5)}{x-4}+\frac{78}{x^2-16}$.

63. $(a^2-b^2)x^2-2(a^2+b^2)x=b^2-a^2$.

64. $\sqrt{2x+9}+\sqrt{3x-15}=\sqrt{7x+8}$.

65. $\frac{1}{x+\sqrt{2-x^2}}+\frac{1}{x-\sqrt{2-x^2}}=\frac{x}{2}$.

66. $\sqrt{\frac{1}{x}}+\sqrt{x}=\sqrt{1+2x}$.

EQUATIONS IN THE QUADRATIC FORM.

468. Any equation is said to be *quadratic in form* when it is composed of three terms, two of which contain the unknown number with an exponent in one term twice its exponent in the other. Thus, the equation $az^{2n} + bz^n = c$ is quadratic in form.

469. Any equation reducible to the form $az^{2n} + bz^n = c$ (1), in which z represents any expression simple or compound, and n any exponent, may be solved, at least in part, by the methods for the solution of complete quadratics: for, denoting z^n by x, and hence z^{2n} by x^2, (1) becomes

$$ax^2 + bx = c;$$

whence
$$x \text{ or } z^n = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

Whether the given equation can be *completely* solved will depend upon the possible solution of the two equations

$$z^n = \frac{-b + \sqrt{b^2 + 4ac}}{2a},$$

$$z^n = \frac{-b - \sqrt{b^2 + 4ac}}{2a}.$$

470. The methods of procedure in solving such equations will be best understood from the following examples.

1. Solve the equation $x^4 + 21x^2 = 100$.

Treating x^2 as the unknown number, we obtain

$$x^2 = -10\tfrac{1}{2} \pm \sqrt{(\tfrac{21}{2})^2 + 100}.$$

$$\therefore\ x^2 = -10\tfrac{1}{2} \pm 14\tfrac{1}{2} = 4 \text{ or } -25.$$

Solving the equations,

$$x^2 = 4,\ x = \pm 2.$$

$$x^2 = -25,\ x = \pm 5\sqrt{-1}.$$

It is thus seen that the given equation has four roots, two being real, and two imaginary; and it may be found by substituting that each root will satisfy the given equation.

2. Solve the equation $3x^6 + 20x^3 = 32$.

Treating x^3 as the unknown number, we obtain

$$x^3 = \frac{-10 \pm \sqrt{100 + 96}}{3} = \frac{-10 \pm 14}{3} = \frac{4}{3} \text{ or } -8.$$

Extracting the cube root, $x = \sqrt[3]{\frac{4}{3}}$ or -2.

This equation has also four imaginary roots, the finding and explaining of which belong to a more advanced treatise.

3. Solve the equation $5\sqrt{x} - 3\sqrt[4]{x} = 14$.

This equation may be presented under the form

$$5x^{\frac{1}{2}} - 3x^{\frac{1}{4}} = 14.$$

Treating $x^{\frac{1}{4}}$ as the unknown number, we obtain

$$x^{\frac{1}{4}} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 70}}{5} = \frac{\frac{3}{2} \pm \frac{17}{2}}{5} = 2 \text{ or } -\frac{7}{5}.$$

Hence $x = (x^{\frac{1}{4}})^4 = 2^4$ or $(-\frac{7}{5})^4 = 16$ or $\frac{2401}{625}$.

4. Solve the equation $\sqrt{x+4} - 5\sqrt[4]{x+4} = -6$.

Changing to $(x+4)^{\frac{1}{2}} - 5(x+4)^{\frac{1}{4}} = -6$, and treating $(x+4)^{\frac{1}{4}}$ as the unknown number, we obtain

$$(x+4)^{\frac{1}{4}} = \tfrac{5}{2} \pm \sqrt{(\tfrac{5}{2})^2 - 6} = \tfrac{5}{2} \pm \tfrac{1}{2} = 3 \text{ or } 2.$$

Hence $x + 4 = [(x+4)^{\frac{1}{4}}]^4 = 3^4$ or $2^4 = 81$ or 16.

$$\therefore x = 77 \text{ or } 12.$$

5. Solve the equation $x^3 = 1$.

Transposing, $x^3 - 1 = 0$;

factoring, $(x-1)(x^2 + x + 1) = 0$;

equating factors to zero, $x - 1 = 0, \ \therefore x = 1$; (1)

$x^2 + x + 1 = 0$. (2)

Solving (2) as a quadratic, $x = \frac{-1 \pm \sqrt{-3}}{2}$.

Hence the equation $x^3 = 1$ has three roots, 1, $\frac{-1+\sqrt{-3}}{2}$, and $\frac{-1-\sqrt{-3}}{2}$, one root being real, and the other two imaginary.

471. Certain equations which are not quadratic in form may be put in the quadratic form by eliminating a factor.

Solve the equations

6. $x^4 - 10x^2 + 9 = 0$.
7. $3x^4 - 76x^2 + 25 = 0$.
8. $x^4 - 6x^2 - 4 = 12$.
9. $x^6 - 2x^3 = 48$.
10. $x^6 - 4x^3 - 28 = 4$.
11. $x^{4n} - 2px^{2n} = q$.
12. $x^4 - 3x^2 = 2\sqrt{x^4 - 3x^2}$.
13. $7x^8 - 12x^4 + 5 = 0$.
14. $x + \sqrt{5x + 10} = 8$.
15. $x + \sqrt{10x + 6} = 9$.
16. $(x^2 + 2)^2 = 2x^4 + 8$.
17. $\sqrt{x+6} + \sqrt{x+3} = 3\sqrt{x}$.
18. $\sqrt{x} + 2 = 2\sqrt{2x+1}$.
19. $x^3 - x^{\frac{3}{2}} = 56$.
20. $x + 5 = \sqrt{x+5} + 6$.
21. $x^6 - 20x^3 = 189$.
22. $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24$.
23. $x^2 + 6\sqrt{x^2 - 2x + 5} = 2x + 11$.
24. $9x + \sqrt{x^2 - 3x + 5} = 3x^2 + 11$.
25. $\sqrt{x+3} + \sqrt[4]{x+3} = 6$.
26. $2(\sqrt{1-x}+1) = \frac{x}{\sqrt{1+x}-1}$.
27. $x^n - 2ax^{\frac{n}{2}} = b^2$.
28. $3x^{\frac{4}{3}} - \frac{5}{2}x^{\frac{2}{3}} = 38$.
29. $\sqrt{a^2 - x^2} + x\sqrt{a^2 - 1} = a\sqrt{1 - x^2}$.
30. $\frac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = \frac{x}{a - x}$.

472. Equations of the third and higher degrees may be readily solved *by factoring*, if, when the second member is made zero, the first member is the product of rational factors.

Take, for example, the equation $x^3 - 7x - 6 = 0$.

Factoring by synthetic division (§ 215),

$$(x+1)(x+2)(x-3)=0;$$

equating factors to zero, $x+1=0,\ x+2=0,\ x-3=0;$

whence $x=-1,\ x=-2,\ x=3.$

Hence the roots are -1, -2, and 3.

Solve by factoring by synthetic division

31. $x^3-10x^2+31x-30=0.$ 33. $x^4-5x^2+4=0.$

32. $x^3-x^2-8x+12=0.$ 34. $x^4-4x^3-7x^2+34x-24=0.$

PROBLEMS INVOLVING QUADRATICS.

1. The perimeter of a rectangular field is 500 yards, and its area 14,400 square yards. What is the length of the sides?

500 yd. $\div 2 = 250$ yd., length of two adjacent sides.

Let $x =$ length of one side;

then $250 - x =$ length of the other side.

Hence, by the conditions,

$$x(250-x)=14400;$$

whence $x^2-250x=-14400.$

Solving equation, $x=125\pm 35=160$ or 90.

If we take $x=160$, the other side is 90, and *vice versa*. Thus, though x has two values, the problem has but one solution.

2. By selling a lot of goods for \$ 24, a merchant lost as many cents on the dollar as he paid dollars for the goods. How much did he pay?

Let $x =$ number of dollars paid;

then $x-24 =$ number of dollars lost.

Hence, by the conditions, $x \times x = (x-24)\,100.$

Simplifying, etc., $x^2-100x=-2400;$

solving equation, $x=60$ or 40.

Either of these values of x satisfies the conditions of the problem: for, if he paid \$ 60 for the goods, he lost 60% by selling them for \$ 24; and, if he paid \$ 40 for the goods, he lost 40% by selling them for \$ 24. It is thus seen that the problem admits of two solutions.

3. A drover bought a number of oxen for $400; but, if he had obtained 4 more for the same money, the price paid per head would have been $5 less. How many oxen did he buy?

Let x = number bought;

then $\frac{400}{x}$ = price per head,

and $\frac{400}{x+4}$ = price per head if 4 more had been bought.

Hence, by the conditions,

$$\frac{400}{x+4} + 5 = \frac{400}{x}.$$

Solving equation, $x = 16$ or -20.

The negative root (-20) is not admissible, since it will not satisfy the conditions of the problem, and hence the number of oxen bought was 16.

4. The sum of the ages of a father and son is 65 years, and the product of their ages is 250 more than 10 times their sum. What is the age of each?

Let x = father's age;

then $65 - x$ = son's age.

Hence $(65 - x)x = 900$.

Solving equation, $x = 45$ or 20.

The father's age is 45 years, and the son's 20 years. Here the second value of x is inadmissible, since it would make the son older than his father.

5. Divide the number 12 into two parts such that their product will be 40.

Let x = one part;

then $12 - x$ = the other part.

Hence $(12 - x)x = 40$.

Solving the equation, $x = 6 \pm 2\sqrt{-1}$.

Both values of x are imaginary, and hence the given problem is impossible.

473. It is shown by the above examples that the solution of problems involving quadratics may give results all of which do not satisfy the conditions of the problem. This is due to the

fact that there may be limitations in the problem, expressed or implied, which do not appear in the equation.

Hence, in solving problems, only those values should be retained as answers which satisfy all the conditions of the given problem. The arithmetical values that will satisfy a problem are *positive*, and usually the first values found.

474. In some cases a change in the wording of a given problem will form an analogous problem, to which the absolute value of the negative root found is an answer. Thus, if Problem 3 above be so changed as to state, that, if 4 *less* oxen had been bought, each would have cost \$ 5 *more*, the values of x would be 20 and -16, the latter being inadmissible.

Imaginary roots indicate that the problem is impossible.

6. The sum of two numbers is 17, and their product 60. What are the numbers?

7. The sum of two numbers is 72, and their product is 10 times as great. What are the numbers?

8. The sum of two numbers is 50, and the sum of their squares is 1282. Find the numbers.

9. The difference of two numbers is 7, and their product is 2340. Find the numbers.

10. The sum of two numbers is m, and their product is n. What are the numbers?

11. The sum of two numbers is m, and the sum of their reciprocals is n. What are the numbers?

12. Find a number such that 5 times its square increased by 10 times the number itself will equal 495.

13. Find three numbers such that the second will be one half of the first, and the third one third of the first, and the sum of their squares will be 441.

14. A certain number of persons pay together a bill of \$ 190, each paying \$ 9 less than the number of persons. How many were there, and how much did each pay?

15. A piece of ground one rod longer than broad contains 1190 square rods. What is the length?

16. An army corps consisting of 12,850 soldiers was formed into two squares, one of which had 10 more men in a side than the other. How many men in each square?

17. A man made a journey of 48 miles in a certain number of hours. If he had traveled 4 miles more per hour, he would have made the journey in 6 hours less time. How many miles per hour did he travel?

18. A farmer bought a number of sheep for \$ 80. If he had bought 4 more sheep for the same money, he would have paid \$ 1 less per head. How many sheep did he buy?

19. A man, being asked his age, replied, "If to the square root of my age you add $\frac{5}{9}$ of my age, the sum will be 26 years." What was his age?

20. Two couriers, A and B, start at the same time to go to a place 90 miles distant. A traveled 1 mile per hour faster than B, and reached the place 1 hour before B. At what rate did each travel?

21. A merchant bought a number of fur robes for \$ 150, and then sold them at \$ 18 a robe, and thus gained on each robe twice the cost of a robe. How many robes did he buy?

22. A farmer has two square fields. The side of one is $2\frac{1}{2}$ rods longer than the side of the other, and the fields together contain $1131\frac{1}{4}$ square rods. How many more square rods in the larger field than in the smaller?

23. A man sold a farm for \$ 3150, and afterwards bought another farm containing 7 more acres for the same money, at \$ 5 less per acre. How many acres in the farm sold, and how much did he receive per acre?

24. A certain number of pieces of cloth cost $1260, each piece costing $9 more than 5 times the number of pieces. How many pieces were there, and how much did each cost?

25. A certain number of persons equally engaged in a business transaction lost $96,000; but, four of them becoming insolvent, each of the rest had to pay $4000 more than his fair share. How many persons were engaged in the business?

26. A jeweler sold a watch for $144, gaining on it as much per cent as the watch had cost him. How much did it cost him?

27. A number of horses were bought for $1800. Had 3 more been obtained for the same money, each would have cost $30 less. How many horses were bought?

28. A cistern supplied by two pipes could be filled by one alone in 5 hours less than by the other alone, and both together could fill it in 6 hours. In how many hours could each fill it alone?

29. A steamer performed its down trip of 150 miles at a certain rate per hour. On the return trip, going 3 miles an hour slower, it took $2\frac{1}{2}$ hours longer. What was the rate down the river?

30. A man bought a certain amount of sugar for $66; but, if sugar were to rise one cent per pound, he would obtain 50 pounds less for the same money. How much sugar did he buy?

31. A drover bought a certain number of sheep for $483. Reserving 20 of the number, he sold the rest for $432, gaining $1 on each. How many sheep did he buy?

32. What is the price of eggs per dozen when 3 less in 25 cents' worth raised the price 5 cents per dozen?

Find a general formula for the above problem, putting a, b, c, for the above numbers.

CHAPTER XVI.

SIMULTANEOUS QUADRATIC EQUATIONS.

475. The solution of simultaneous equations of the second degree with two unknown numbers involves the elimination of one of the unknown numbers, and the forming of an equation with but one unknown number. This may lead to an equation of a higher degree than the second, usually of the fourth, which cannot in general be solved by quadratic methods.

476. There are, however, several cases in which the solution of such simultaneous equations can be effected by equations of the second degree. The three cases of most frequent occurrence are

I. *When one of the given equations is simple.*

II. *When the equations are homogeneous and quadratic.*

III. *When the equations are symmetrical with respect to the unknown numbers.*

A few solutions will sufficiently illustrate the process in each case.

477. I. One of the given equations simple.

1. Solve the equations $\begin{cases} x^2 + 3xy - y^2 = 29, & (1) \\ 4x - y = 7. & (2) \end{cases}$

From (2), transposing, $y = 4x - 7$; (3)

substituting value of y in (1), $x^2 + 3x(4x-7) - (4x-7)^2 = 29$;

whence $3x^2 - 35x = -78.$ (4)

Solving (4), $x = 8\frac{2}{3}$ or 3;

substituting value of x in (2), $y = 27\frac{2}{3}$ or 5;

$\therefore x = 8\frac{2}{3}, y = 27\frac{2}{3}$; or $x = 3, y = 5.$

If preferred, the values of x and y may be braced in corresponding pairs, thus: $\begin{cases} x = 3, & y = 5, \\ x = 8\frac{1}{3}, & y = 27\frac{1}{3}. \end{cases}$

478. In like manner simultaneous quadratic equations, when one is simple, may be solved *by finding in the simple equation the value of one of the unknown numbers in terms of the other, and substituting in the other equation.*

Solve the following groups of equations:

2. $\begin{cases} x^2 + y^2 = 89, \\ x - y = 3. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 16, \\ 2x + y = 13. \end{cases}$

4. $\begin{cases} x^2 - 7y^2 = 1, \\ x - 2y = 2. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 9, \\ 2x + y = 14. \end{cases}$

6. $\begin{cases} 5x^2 - 2y^2 = 93, \\ 3x - 4y = -1. \end{cases}$

7. $\begin{cases} x^2 + 2y^2 = 9, \\ x + 2y = 5. \end{cases}$

8. $\begin{cases} 5x^2 - 3xy + y^2 = 45, \\ 3x + 2y = 22. \end{cases}$

9. $\begin{cases} 10x + y = 3xy, \\ y - x = 2. \end{cases}$

10. $\begin{cases} y^2 + 2x^2 = 11, \\ 2y + 3x = 9. \end{cases}$

11. $\begin{cases} 5x^2 - 3xy + y^2 = 2x - 3y + 31, \\ 5x - 2y = 11. \end{cases}$

12. $\begin{cases} (x-2)^2 + (y-6)^2 = 69 - 2xy, \\ 5x - 4y = 1. \end{cases}$

479. II. The equations homogeneous and quadratic.

An equation is said to be **homogeneous** when all its terms which contain the unknown number are homogeneous (§ 56).

13. Solve the equations $\begin{cases} x^2 + xy = 10, & (1) \\ 2xy - y^2 = 3. & (2) \end{cases}$

Multiplying the first member of (1) by the second member of (2), and the first member of (2) by the second member of (1), we have

$$3x^2 + 3xy = 20xy - 10y^2; \quad (3)$$

$$\text{or } 3x^2 - 17yx = -10y^2. \quad (4)$$

Considering $17y$ as the coefficient of x in (4), and solving the equation as a quadratic, we have

$$x = 5y \text{ or } \tfrac{2}{3}y;$$

whence $$2xy = 10y^2 \text{ or } \tfrac{4}{3}y^2.$$

Substituting values of $2xy$ in (2), $9y^2 = 3$; $\tfrac{4}{3}y^2 - y^2 = \tfrac{1}{3}y^2 = 3$;

whence $$y = \pm\tfrac{1}{3}\sqrt{3};\ y = \pm 3.$$

$$\therefore\ x = \pm\tfrac{5}{3}\sqrt{3};\ x = \pm 2.$$

In like manner any two homogeneous equations of the second degree may be solved.

480. Two homogeneous simultaneous equations of the second degree may also be solved by assuming $y = vx$, and substituting vx for y in both equations, and then by division or otherwise obtaining an equation involving only v. Having found the value of v in this equation, the values of x and y may be found by substitution. It is, however, believed that the method illustrated by Example 13 is simpler.

Solve the following groups of equations:

14. $\begin{cases} x^2 - y^2 = 5, \\ 4x^2 - 3xy = 18. \end{cases}$

15. $\begin{cases} 2y^2 - xy = 10 - 4x^2, \\ 2x^2 + 3xy = 2y^2. \end{cases}$

16. $\begin{cases} 2y^2 + xy = 5, \\ x^2 - 2xy = y^2 + 2. \end{cases}$

17. $\begin{cases} x^2 - 2y^2 = 8, \\ 3y^2 - xy = 4. \end{cases}$

18. $\begin{cases} x^2 + y^2 = 61, \\ x^2 - xy = 6. \end{cases}$

19. $\begin{cases} 3x^2 - 7y^2 + 1 = 0, \\ xy = 6. \end{cases}$

20. $\begin{cases} 2xy + 24 = 3x^2, \\ y^2 - xy = -3. \end{cases}$

21. $\begin{cases} x^2 - 2xy = 9 - 3y^2, \\ x^2 - 4xy = 5 - 5y^2. \end{cases}$

22. $\begin{cases} 2x^2 - 3xy = 4 - y^2, \\ x^2 - 2xy = 9 - 3y^2. \end{cases}$

23. $\begin{cases} x(x-y) + y(x-y) = 158, \\ 7x(x+y) = 72y(x-y). \end{cases}$

481. III. The equations symmetrical.

An equation is said to be **symmetrical** with respect to two numbers, as x and y, if the numbers can be interchanged without destroying the equality.

Thus, $x^2 - 2xy + y^2 = 12$ is symmetrical; for, on interchanging x and y, it becomes $y^2 - 2yx + x^2 = 12$, a true equation, since $x^2 - 2xy + y^2 = y^2 - 2yx + x^2$ (§ 80). The equation $x^2 - y^2 = 16$ is not symmetrical, since $x^2 - y^2$ is not equivalent to $y^2 - x^2$.

24. Solve the equation $\begin{cases} x^2 + y^2 = 74, & (1) \\ x + y = 12. & (2) \end{cases}$

Squaring (2), $x^2 + 2xy + y^2 = 144$; (3)

subtracting (1) from (3), $2xy = 70$; (4)

subtracting (4) from (1), $x^2 - 2xy + y^2 = 4$;

extracting square root, $x - y = \pm 2$; (5)

adding (5) and (2), $\begin{cases} 2x = 14, \therefore x = 7; \\ 2x = 10, \therefore x = 5. \end{cases}$

Substituting 7 for x in (2), $y = 5$;

substituting 5 for x in (2), $y = 7$.

Hence the values of x and y are $x = 7$, $y = 5$; $x = 5$, $y = 7$.

482. In like manner any two simultaneous symmetrical equations of the second degree may be solved *by so combining them as to obtain the values of the sum and the difference of the unknown numbers.*

483. Groups of equations which are symmetrical except in the signs of the terms may often be solved by the symmetrical method.

25. Solve the equations $\begin{cases} x^2 + y^2 = 52, & (1) \\ x - y = -2. & (2) \end{cases}$

Squaring (2), $x^2 - 2xy + y^2 = 4$; (3)

subtracting (3) from (1), $2xy = 48$; (4)

adding (1) and (4), $x^2 + 2xy + y^2 = 100$;

extracting square root, $x + y = \pm 10$; (5)

adding (2) and (5), $2x = 8$ or -12, $\therefore x = 4$ or -6;

subtracting (2) from (5), $2y = 12$ or -8, $\therefore y = 6$ or -4.

$\therefore x = 4, y = 6$; or $x = -6, y = -4$.

26. Solve the equations $\begin{cases} x^3+y^3=35, & (1) \\ x+y=5. & (2) \end{cases}$

Dividing (1) by (2), $x^2-xy+y^2=7$; (3)

squaring both members of (2), $x^2+2xy+y^2=25$; (4)

subtracting (3) from (4), $3xy=18$;

whence $xy=6$. (5)

Subtracting (5) from (3), $x^2-2xy+y^2=1$;

extracting square root, $x-y=\pm 1$; (6)

adding (2) and (6), $2x=6$ or 4, $\therefore x=3$ or 2.

Substituting 3 or 2 for x in (2), $y=2$ or 3.

484. Groups of symmetrical equations often fall under the first case, and are readily solved by substitution.

Solve the following groups of equations:

27. $\begin{cases} x^2+y^2=74, \\ xy=35. \end{cases}$

28. $\begin{cases} x+y=2, \\ xy=-15. \end{cases}$

29. $\begin{cases} x^2+y^2=61, \\ x+y=-1. \end{cases}$

30. $\begin{cases} x+y=5, \\ x^2y+xy^2=30. \end{cases}$

31. $\begin{cases} x^2+4y^2=25, \\ 2xy=12. \end{cases}$

32. $\begin{cases} x^2+y^2=29, \\ x-y=3. \end{cases}$

33. $\begin{cases} x^2+xy+y^2=37, \\ x+y=7. \end{cases}$

34. $\begin{cases} x^2-y^2=39, \\ x-y=3. \end{cases}$

35. $\begin{cases} x^2-xy+y^2=19, \\ x-y=3. \end{cases}$

36. $\begin{cases} x^3-y^3=37, \\ x-y=1. \end{cases}$

37. $\begin{cases} x^3+y^3=91, \\ x+y=1. \end{cases}$

38. $\begin{cases} x^2+xy+y^3=175, \\ x^3-y^3=875. \end{cases}$

SPECIAL METHODS.

485. The preceding methods of solving simultaneous equations of the second degree are called *general* because they apply to classes of equations. There are, however, many simultaneous equations not falling under these cases, which are readily solved by special artifices. Some equations that do fall under these cases may be solved more elegantly by special methods.

1. Solve the equations $\begin{cases} x^4 - y^4 = 369, & (1) \\ x^2 + y^2 = 41. & (2) \end{cases}$

Dividing (1) by (2), $x^2 - y^2 = 9$; (3)

adding (2) and (3), $2x^2 = 50$; $\therefore x = \pm 5$.

Subtracting (3) from (2), $2y^2 = 32$; $\therefore y = \pm 4$.

Hence $x = 5,\ y = 4$; or $x = -5,\ y = -4$.

This solution gives only the finite roots of equations (1) and (2).

2. Solve the equations $\begin{cases} x^4 + y^4 = 97, & (1) \\ x + y = 5. & (2) \end{cases}$

Raising (2) to the 4th power, we have

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625; \quad (3)$$

adding (1) and (3), and dividing by 2,

$$x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = 361;$$

extracting the square root, $x^2 + xy + y^2 = \pm 19$. (4)

We now have two equations, (2) and (4), which can be readily solved by substitution or by the symmetrical method.

3. Solve the equations

$$\begin{cases} x^2 + 2xy + x + 3y = 75, & (1) \\ y^2 + x - y = 24. & (2) \end{cases}$$

Adding (1) and (2), $x^2 + 2xy + y^2 + 2x + 2y = 99$; (3)

factoring parts and adding 1, $(x + y)^2 + 2(x + y) + 1 = 100$;

extracting square root, $x + y + 1 = \pm 10$.

$\therefore x + y = 9$ or -11. (4)

Equations (2) and (4) can now be solved by substitution, the first general method.

Solve the following groups of equations:

4. $\begin{cases} x^2 - y^2 = 16, \\ x + y = 8. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 32, \\ x - y = 2. \end{cases}$

6. $\begin{cases} x^3 - y^3 = 19, \\ x - y = 1. \end{cases}$

7. $\begin{cases} x^3 + y^3 = 468, \\ x + y = 12. \end{cases}$

8. $\begin{cases} x^4 - y^4 = 544, \\ x^2 + y^2 = 34. \end{cases}$

9. $\begin{cases} x^4 - y^4 = 1280, \\ x^2 - y^2 = 32. \end{cases}$

10. $\begin{cases} x^4 + y^4 = 706, \\ x + y = 8. \end{cases}$

11. $\begin{cases} x^2 - 3xy + y^2 = 59, \\ x^2 + y^2 = 29. \end{cases}$

12. $\begin{cases} xy + 1 = x + y, \\ xy = 2. \end{cases}$

13. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

14. $\begin{cases} x^4 + y^4 = 272, \\ x - y = 2. \end{cases}$

15. $\begin{cases} x^4 + y^4 = 2657, \\ x + y = 11. \end{cases}$

16. $\begin{cases} x^2 + 2xy + y + 3x = 73, \\ y^2 + x + 3y = 44. \end{cases}$

17. $\begin{cases} x^2 + xy = a^2 + ab, \\ y^2 + yx = b^2 + ab. \end{cases}$

PROBLEMS INVOLVING SIMULTANEOUS QUADRATICS.

486. The statement of problems involving two or more unknown numbers has been sufficiently illustrated under the head of simple equations. Several of the problems below can be solved by the use of only one unknown number, but their solution is facilitated by the use of two.

1. The difference of two numbers is 7, and the difference of their squares is 119. What are the numbers?

2. The sum of two numbers is 18, and the sum of their squares is 170. What are the numbers?

3. The sum of two numbers is 25, and the difference of their squares is 175. What are the numbers?

4. Find two numbers such that the first increased by twice the second is 24, and the sum of their squares is 149.

5. The sum of the squares of two numbers exceeds twice their product by 9, and the difference of their squares is 1 less than their product. Find the numbers.

6. The sum of six times the greater of two numbers and five times the less is 50, and their product is 20. Find the numbers.

7. The product of two numbers diminished by their sum is 17, and the sum of their squares is 65. Find the numbers.

8. Find two numbers such that their sum is 19, and the sum of their cubes 1843.

9. Find two numbers such that their difference is 4, and the difference of their cubes 988.

10. The product of two numbers multiplied by their sum is 180, and the sum of their cubes is 189. Find the numbers.

11. If a certain number expressed by two digits be multiplied by the sum of its digits, the product will be 160; and, if the number be divided by four times its unit digit, the quotient will be 4. Find the number.

12. What number divided by the product of its two digits is $5\frac{1}{3}$, but, when 9 is subtracted from it, the resulting number is expressed by the two digits in an inverse order?

13. The area of a rectangular field is 300 square rods, and the length of its diagonal is 25 rods. Find the length of the sides.

14. The sum of the diagonal and the longer side of a rectangle is three times the length of the shorter side, and the difference in the lengths of the two sides is 4 yards. What is the area of the rectangle?

15. The area of the floor of a certain hall is 5375 sq. ft., and its length is 4 feet less than three times its breadth. What are the dimensions of the floor?

16. A certain number of sheep were bought for $468; but, after 8 of them had been reserved, the rest were sold at an advance of $1 a head, and $12 were gained on the lot. How many sheep were bought?

17. A vessel can be filled in 6 hours by two pipes running at the same time, but one pipe can fill it alone in 5 hours less than the other. How many hours does each pipe require to fill it?

CHAPTER XVII.

INEQUALITIES.

487. The expression $a > b$ denotes that a is greater than b, and $a < b$ denotes that a is less than b (§ 39). The sign $>$ or $<$ is called the **sign of inequality.**

488. An **inequality** is an expression consisting of two unequal numbers connected by the sign of inequality. Thus $4 > 3$ and $x < y$ are inequalities.

489. Two inequalities are said to subsist in the *same sense* when their first members are both greater or both less than their second members. Thus, $a > b$ and $c > d$ subsist in the same sense.

490. Two inequalities are said to subsist in a *contrary sense* when the first member is the greater in one, and the less in the other. Thus, $a > b$ and $c < d$ subsist in a contrary sense.

Inequalities are also called *inequations;* and two inequalities which subsist in the same sense are also said to be of the *same direction*, since the signs point in the same direction ; and two inequalities that subsist in a contrary sense are also said to be the *reverse*, since the signs point in opposite directions.

It is assumed in this chapter, unless the contrary be stated, that the letters denote real and positive numbers.

491. $5 > 3$, and $5 + 2 > 3 + 2$; and, generally, if $a > b$, then $a + c > b + c$. Likewise $5 > 3$, and $5 - 2 > 3 - 2$; and, generally, if $a > b$, then $a - c > b - c$. Hence, *if the same positive number be added to or subtracted from both members of an inequality, the resulting inequality will subsist in the same sense.*

492. It follows that a term can be transposed from one member of an inequality to the other, as in an equation, *provided its sign be changed.*

493. If $a > b$, then $2a > 2b$, $3a > 3b$, and $ac > bc$; and, if $a > b$, then $\frac{a}{2} > \frac{b}{2}$, $\frac{a}{3} > \frac{b}{3}$, and $\frac{a}{c} > \frac{b}{c}$. Hence, *if both members of an inequality be multiplied or divided by the same positive number, the resulting inequality will subsist in the same sense.*

494. A positive number is greater, algebraically considered, than any negative number; and, of two unequal negative numbers, the less numerically considered is the greater algebraically (§ 67). Thus, $2 > -7$, and $-2 > -7$.

Hence, *if both members of an inequality be multiplied or divided by the same negative number, the resulting inequality will subsist in a contrary sense.* Thus, $2 < 5$, but $2 \times (-3) > 5 \times (-3)$; also $-2 > -5$, but $-2 \times (-3) < -5 \times (-3)$.

495. It follows, that, *if the signs of both members of an inequality be changed, the resulting inequality will subsist in a contrary sense.*

496. If $a > b$, $c > d$, and $e > f$, then $a + c + e > b + d + f$, and $a \times c \times e > b \times d \times f$. Hence, *if two or more inequalities that subsist in the same sense be added member to member, or multiplied member by member, the resulting inequality will subsist in the same sense.*

497. If an inequality be subtracted from, or divided by, another inequality in the same sense, the resulting inequality may or may not subsist in the same sense. Thus, $5 < 8$, and $1 < 5$, but $5 - 1 > 8 - 5$; $4 < 6$, and $1 < 3$, but $4 \div 1 > 6 \div 3$.

These operations are to be avoided when the sense of the resulting inequality cannot be determined.

498. If $a > b$, then $a^2 > b^2$, $a^3 > b^3$, and $a^n > b^n$. Hence, *if both members of an inequality of positive numbers be raised to the same power, the resulting inequality will subsist in the same sense.*

499. If the members of an inequality are *negative*, the same odd powers will subsist in the same sense, and the same even powers in a contrary sense. Thus, if $-a > -b$, then $(-a)^3 > (-b)^3$, but $(-a)^2 < (-b)^2$.

500. If $a^4 > b^4$, then $a^2 > b^2$ and $a > b$. Hence, *if the same root of both members of an inequality of positive numbers be taken, the resulting inequality will subsist in the same sense.*

501. An inequality, like an equation, may be simplified by certain transformations; and an inequality is said to be solved *when a limit to the value of the unknown number is found.*

1. Simplify the inequality $\frac{3x}{2} - 2 > 3 + \frac{2x}{3}$.

Transposing -2 and $\frac{2x}{3}$, $\quad \frac{3x}{2} - \frac{2x}{3} > 5$;

clearing of fractions, and uniting, $\quad 5x > 30$;

dividing by 5, $\quad x > 6$.

Hence 6 is a limit of the value of x.

2. Simplify $2x + \frac{x}{3} - 4 > \frac{3x}{2} - \frac{2}{3}$.

3. Find the limits of x, when given $\begin{cases} \frac{x}{b} + b < \frac{x}{a} + a, & (1) \\ \frac{x}{b} - \frac{x}{a} > \frac{a-b}{b}. & (2) \end{cases}$

Transposing in (1), $\quad \frac{x}{b} - \frac{x}{a} < a - b$;

clearing of fractions, $\quad ax - bx < a^2b - ab^2$;

factoring, $\quad (a-b)x < (a-b)ab$;

dividing by $a - b$, $\quad x < ab$.

Clearing (2) of fractions, $\quad ax - bx > a^2 - ab$;

factoring and dividing by $a - b$, $\quad x > a$.

Hence the limits of x are a and ab.

4. Find the limit of x in $\frac{2x}{b} - \frac{a}{x} < \frac{x^2 - a}{bx}$.

Find the limits of x in

5. $\begin{cases} x-5>10-2x, \\ 10x<3x+49. \end{cases}$

6. $\begin{cases} x+5>\frac{x}{3}+55, \\ \frac{x}{2}+2<\frac{x}{3}+15. \end{cases}$

7. $\begin{cases} \frac{4x-2}{3}>\frac{2-4x}{5}, \\ 3x-2<\frac{x}{2}+5\frac{1}{3}. \end{cases}$

8. $\begin{cases} \frac{x}{a-b}>\frac{a-b}{x}, \\ ax-bx<\frac{a^2-b^2}{x}. \end{cases}$

9. $\begin{cases} ax+bx>ab+b^2, \\ \frac{ax}{2}-bx<\frac{a^2}{2}-b. \end{cases}$

10. $\begin{cases} \frac{x}{a}+a<\frac{x}{b}+b, \\ \frac{x}{a}+\frac{x}{b}>\frac{a+b}{a}. \end{cases}$

502. *The sum of the squares of any two unequal numbers is greater than twice the product of the numbers.*

For let a and b be any two unequal numbers. Then, since $(a-b)^2$ is positive whether $a>$ or $<b$, we have

$$(a-b)^2>0.$$

Expanding, $$a^2-2ab+b^2>0;$$

transposing $-2ab$, $$a^2+b^2>2ab.$$

11. Show that the sum of any fraction whose terms are unequal and its reciprocal is greater than 2.

Let $\frac{a}{b}$ be any fraction in which $a>$ or $<b$. Then, by § 502, we may assume that $$a^2+b^2>2ab.$$

Dividing by ab, $$\frac{a}{b}+\frac{b}{a}>2.$$

Assume that a and b are positive unequal numbers, and show that

12. $a^3b+ab^3>2a^2b^2$.

13. $a^3+b^3>a^2b+ab^2$.

14. $\frac{a+b}{2}>\frac{2ab}{a+b}$.

CHAPTER XVIII.

RATIO, PROPORTION, VARIATION.

RATIO.

503. **Ratio** is the relation of one number to another of the same kind expressed by their quotient. Thus, the ratio of a to b is $\frac{a}{b}$.

Every fraction expresses the ratio of its numerator to its denominator; and every integer expresses the ratio of itself to unity.

504. A ratio may be expressed by writing a colon (:) between its two terms. Thus, the ratio of a to b is expressed by $a : b$, read "the ratio of a to b," or, briefly, "a to b."

The ratio of a to b is expressed by $a : b$, and the ratio of b to a by $b : a$; but the ratio *between* a and b or b and a is expressed by $a : b$ or $b : a$. Thus, the ratio of 3 to 6 is $\frac{3}{6}$, and the ratio of 6 to 3 is $\frac{6}{3}$; but the ratio *between* 3 and 6 or 6 and 3 is $\frac{3}{6}$ or $\frac{6}{3}$.

505. The first term of a ratio is called the **antecedent**; and the second term, the **consequent**. The two terms of a ratio taken together are called a **couplet**.

The antecedent of a ratio is the dividend; and the consequent, the divisor.

506. When the antecedent equals the consequent, as $a : a$, the ratio equals unity, and is called the **ratio of equality**.

When the antecedent is greater than the consequent, as $8 : 5$, the ratio is greater than unity, and is called a **ratio of greater inequality**.

When the antecedent is less than the consequent, as $5:8$, the ratio is less than unity, and is called a **ratio of less inequality.**

When the antecedent and the consequent are interchanged, the resulting ratio is the **inverse** of the given ratio. Thus, $b:a$ is the inverse of $a:b$.

507. Since $\frac{a}{b}=\frac{a\times n}{b\times n}$, and $\frac{a}{b}=\frac{a\div n}{b\div n}$, *both terms of a ratio may be multiplied or divided by the same number without altering the value of the ratio.*

508. If $\frac{a}{b}=r$, $\frac{a\times n}{b}=rn$, and $\frac{a}{b\div n}=rn$; hence multiplying the antecedent or dividing the consequent of a ratio by a number *multiplies* the ratio by that number.

509. If $\frac{a}{b}=r$, $\frac{a\div n}{b}$ or $\frac{a}{b\times n}=\frac{r}{n}$; hence dividing the antecedent or multiplying the consequent of a ratio by a number *divides* the ratio by that number.

510. Ratios may be compared *by reducing the fractions that express them to a common denominator, and comparing the resulting numerators.*

Thus, to compare $a:b$ with $c:d$, we reduce $\frac{a}{b}$ and $\frac{c}{d}$ to $\frac{ad}{bd}$ and $\frac{bc}{bd}$ respectively, and then decide that $a:b >$ or $=$ or $< c:d$, according as $ad >$ or $=$ or $< bc$.

511. If $a>b$, $\frac{a+n}{b+n}<\frac{a}{b}$; and if $a<b$, $\frac{a+n}{b+n}>\frac{a}{b}$.

For, reducing the ratios to a common denominator, we have

$$\frac{a+n}{b+n}=\frac{ab+bn}{b(b+n)}, \text{ and } \frac{a}{b}=\frac{ab+an}{b(b+n)}.$$

If $a>b$, $(ab+bn)<(ab+an)$, and hence $\frac{a+n}{b+n}<\frac{a}{b}$; but, if $a<b$, $(ab+bn)>(ab+an)$, and hence $\frac{a+n}{b+n}>\frac{a}{b}$.

Hence a ratio of greater inequality is diminished, and a ratio of less inequality increased, *by adding the same number to both terms of the ratio.*

512. If $a > b$, $\frac{a-n}{b-n} > \frac{a}{b}$; and, if $a < b$, $\frac{a-n}{b-n} < \frac{a}{b}$.

For, reducing the ratios to a common denominator, we have

$$\frac{a-n}{b-n} = \frac{ab-bn}{b(b-n)}, \text{ and } \frac{a}{b} = \frac{ab-an}{b(b-n)}.$$

If $a > b$, $(ab-bn) > (ab-an)$, and hence $\frac{a-n}{b-n} > \frac{a}{b}$; but, if $a < b$, $(ab-bn) < (ab-an)$, and hence $\frac{a-n}{b-n} < \frac{a}{b}$.

Hence a ratio of greater inequality is increased, and a ratio of less inequality diminished, *by subtracting the same number from both its terms.*

The principles stated in §§ 511, 512, may be thus illustrated:

(1) $\frac{4+2}{3+2} < \frac{4}{3}$, and $\frac{3+2}{4+2} > \frac{3}{4}$; (2) $\frac{4-2}{3-2} > \frac{4}{3}$, and $\frac{3-2}{4-2} < \frac{3}{4}$.

513. Generally, since every ratio may be expressed as a fraction, *whatever operations on the terms of a fraction affect its value, will in like manner affect the value of the corresponding ratio.*

514. A **compound ratio** is the product of two or more ratios. Thus, the ratio $ac : bd$ is the product of $a : b$ and $c : d$, and is called compound.

Two or more ratios may be compounded by taking the product of the fractions that express them. For example, the ratios of $a : b$, $c : d$, and $e : f$, are compounded by taking the product of

$$\frac{a}{b}, \frac{c}{d}, \text{ and } \frac{e}{f}. \quad \text{Thus, } \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}.$$

515. A ratio, as $a : b$, may be compounded with itself. $a^2 : b^2$

is called the *duplicate* of the ratio $a : b$; $a^2 : b^2$, the *triplicate* of $a : b$; and so on.

The ratio $\sqrt{a} : \sqrt{b}$ or $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the *subduplicate* of $a : b$; and $\sqrt[3]{a} : \sqrt[3]{b}$ or $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, the *subtriplicate* of $a : b$.

516. When one or both terms of a ratio are incommensurable (§ 371), the ratio is said to be *incommensurable*. Thus, $1 : \sqrt{2}$ is an incommensurable ratio.

PROBLEMS.

1. What is the ratio of 2 lb. to 2 oz.? Of \$0.75 to \$3?

2. Arrange in descending order of magnitude

$$4 : 5,\ 7 : 3,\ 12 : 4,\ 3 : 8,\ 5 : 12,\ 7 : 5.$$

3. What is the ratio compounded of $2 : 3$ and $15 : 16$? Of $7 : 6$ and $24 : 35$?

4. Two numbers are in the ratio of 3 to 5; but, if each be increased by unity, their ratio becomes $11 : 18$. Find the numbers.

5. Two numbers are in the ratio of 8 to 7; but, if each be increased by 6, their ratio becomes $6 : 5$. Find the numbers.

6. Give the duplicate and triplicate ratios of $5 : 7$, the subduplicate ratio of $81 : 25$, and the subtriplicate ratio of $729 : 343$.

7. Reduce the following ratios to their lowest terms:

$$63 : 45,\ 138 : 124,\ a^2 : ax,\ m^2 - n^2 : m^3 + n^3.$$

8. Find two numbers in the ratio of 4 to 5, such that their difference is to the difference of their squares as 1 to 27.

9. Find two numbers such that the ratio of their sum to the sum of their squares will be as 11 to 195, the ratio of the numbers themselves being $4 : 7$.

10. Find x so that the ratio of x to 1 may be the duplicate of the ratio of 8 to x.

PROPORTION.

517. A **proportion** is the expressed equality of two ratios.

This equality may be expressed by the sign : : or the sign =. Thus, the equality of the two ratios $a : b$ and $c : d$ may be expressed by $a : b : : c : d$ or by $a : b = c : d$, each being read "the ratio of a to b equals the ratio of c to d," or, briefly, "a is to b as c is to d."

518. Since each ratio has two terms (an antecedent and a consequent), a proportion necessarily consists of *four* terms, the first and third being antecedents, and the second and fourth, consequents.

The first and fourth terms of a proportion are called the **extremes**; and the second and third, the **means**. Thus, in $a : b : : c : d$, a and d are the extremes; and b and c, the means.

519. Four numbers are said to be **proportional**, or **in proportion**, when the ratio of the first to the second equals the ratio of the third to the fourth. Thus, a, b, c, d, are proportional when $a : b = c : d$. The four terms of a proportion are called **proportionals**, the fourth term being called the **fourth proportional**.

520. Numbers are in **continued proportion** when the ratios of the first to the second, the second to the third, the third to the fourth, etc., are equal. Thus, a, b, c, d, e, etc., are in continued proportion when $a : b = b : c = c : d = d : e$, etc.

521. Since the ratio $a : b$ may be expressed by the fraction $\frac{a}{b}$, and $c : d$ by $\frac{c}{d}$, the proportion $a : b = c : d$ is identical with the equation $\frac{a}{b} = \frac{c}{d}$; and, by simple transformations of this fundamental equation, the following propositions in proportion are proved.

Propositions.

522. *If four numbers are in proportion, the product of the extremes equals the product of the means.*

Let $a : b = c : d$;

then $\frac{a}{b} = \frac{c}{d}$.

Multiplying by bd, $ad = bc$.

523. It follows from the above, that, *if any three terms of a proportion are given, the other term can be found.*

For, if $ad = bc$, then $a = \frac{bc}{d}$, $d = \frac{bc}{a}$; $b = \frac{ad}{c}$, $c = \frac{ad}{b}$: and hence either extreme of a proportion may be found *by dividing the product of the means by the other extreme;* and either mean may be found *by dividing the product of the extremes by the other mean.*

Find the value of x in the following proportions:

1. $5 : 8 :: 15 : x$.
2. $x : 5 :: 6 : 10$.
3. $5 : x :: 7 : 10$.
4. $4 : 6 :: x : 4$.
5. $x : 10 + x :: 6 : 9$.
6. $x : 15 - x :: 30 : 15$.
7. $6 : x :: 24 : x + 6$.
8. $4 : 18 :: x - 3 : x + 4$.

524. *If the product of two numbers is equal to the product of two other numbers, the four numbers are proportionals; and any two of them may be made the extremes, and the other two the means, of a proportion.*

Let $ad = bc$.

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$;

whence $\frac{a}{b} = \frac{c}{d}$; i.e., $a : b = c : d$.

The equation $ad = bc$, if divided successively by cd, ac, and dc, gives

$$\frac{a}{c} = \frac{b}{d};\ \text{i.e., } a : c = b : d;$$

$$\frac{b}{a} = \frac{d}{c};\ \text{i.e., } b : a = d : c;$$

$$\frac{b}{d} = \frac{a}{c};\ \text{i.e., } b : d = a : c.$$

Let the pupil obtain these proportions from $ad = bc$, and compare them with the next two propositions.

Let the pupil illustrate the above, and also each of the following eight propositions, by means of numbers.

525. *If four numbers are in proportion, they will be in proportion taken alternately; i.e., the first term will be to the third as the second term to the fourth.*

Let $a : b = c : d;$

then $\frac{a}{b} = \frac{c}{d}.$

Multiplying by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d};$ i.e., $a : c = b : d.$

526. *If four numbers are in proportion, they will be in proportion taken inversely; i.e., the second term will be to the first as the fourth term to the third.*

Let $a : b = c : d;$

then $\frac{a}{b} = \frac{c}{d}.$

Dividing 1 by each member, $1 \div \frac{a}{b} = 1 \div \frac{c}{d};$

whence $\frac{b}{a} = \frac{d}{c};$ i.e., $b : a = d : c.$

527. When $a + b$ is to b as $c + d$ is to d, the numbers $a, b, c,$ and d are said to be in proportion *by composition.*

When $a-b$ is to b as $c-d$ is to d, the numbers a, b, c, and d are said to be in proportion *by division.*

When $a+b$ is to $a-b$ as $c+d$ is to $c-d$, the numbers a, b, c, and d are said to be in proportion *by composition and division.*

528. *If four numbers are in proportion, they are in proportion by composition or division.*

Let $$a:b=c:d;$$

then $$\frac{a}{b}=\frac{c}{d}.$$

Adding ± 1 to each member, $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$;

whence $$\frac{a\pm b}{b}=\frac{c\pm d}{d};$$

that is, $$a\pm b:b=c\pm d:d.$$

529. *If four numbers are in proportion, they are in proportion by composition and division.*

Let $$a:b=c:d;$$

then (§ 528) $$\frac{a+b}{b}=\frac{c+d}{d}, \qquad (1)$$

and $$\frac{a-b}{b}=\frac{c-d}{d}. \qquad (2)$$

Dividing (1) by (2), member by member,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d};$$

that is, $$a+b:a-b=c+d:c-d.$$

530. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $$\frac{a}{b}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h};$$

then, by § 522, $ab=ba$, $ad=bc$, $af=be$, $ah=bg$.

Adding and factoring, $a(b+d+f+h)=b(a+c+e+g)$;

that is, $$a+c+e+g:b+d+f+h=a:b.$$

531. *The product of the corresponding terms of two or more proportions are in proportion.*

Let $a:b=c:d,$

and $e:f=g:h;$

or $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}.$

Multiplying member by member, $\frac{ae}{bf}=\frac{cg}{dh}$;

that is, $ae:bf=cg:dh.$

532. *If* a, b, c, d, *are in continued proportion* (§ 520), a *is to* c *as* a^2 *is to* b^2, *and* a *is to* d *as* a^3 *is to* b^3.

Let $a:b=b:c=c:d,$

or $\frac{a}{b}=\frac{b}{c}=\frac{c}{d};$

whence $\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b};$

or $\frac{a}{c}=\frac{a^2}{b^2}$; i.e., $a:c=a^2:b^2.$

Also $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b};$

or $\frac{a}{d}=\frac{a^3}{b^3}$; i.e., $a:d=a^3:b^3.$

533. When three numbers, a, b, c, are in continued proportion, b is called a **mean proportional** to a and c, and c a **third proportional** to a and b.

534. *The mean proportional to two numbers is equal to the square root of their product.*

Let $a:b=b:c;$

then $\frac{a}{b}=\frac{b}{c}.$

Clearing of fractions, $b^2=ac;$

whence $b=\sqrt{ac}.$

535. Several of the foregoing theorems are useful in the solution of certain numerical problems, and also certain equations.

536. Prove the following propositions, and illustrate each with numbers:

I. If two proportions have the same couplet in each, the other couplets will form a proportion.

II. If two proportions have the same antecedents, the consequents are in proportion.

III. If three numbers are in proportion, the ratio of the first to the third is the duplicate ratio of the first to the second.

IV. If the first two terms of a proportion be multiplied by m, and the last two terms by n, the resulting products will be in proportion.

Find the value of x in the following proportions:

1. $x+1 : x+6 = x+17 : x+19$.
2. $3x+3 : 3x-4 = 5x+1 : 5x-3$.
3. $x+a : x+b = x+c : x+d$.
4. $mx+a : qx+b = mx+c : qx+d$.
5. $(x+7)(x-4) : (x+3)(x-1) = (x+1)(x-4) : (x+1)(x-5)$.

PROBLEMS.

1. Find a fourth proportional to 4, 6, 12.
2. Find a fourth proportional to $\frac{2}{3}$, $\frac{8}{5}$, $\frac{5}{6}$.
3. Find a mean proportional to 4 and 9; 4 and 16.
4. Find a third proportional to 9 and 12; 7 and 14.
5. Find a mean proportional to $\frac{2}{5}$ and $\frac{8}{27}$; $\frac{3}{5}$ and 30.
6. Find a third proportional to a^2 and $2ab$; xy and $3xy^2$.
7. Divide the number 20 into two parts such that the ratio of their squares will be as 9 to 4.
8. Divide the number a into two parts such that the ratio of their squares will be as m^2 to n^2.

9. Divide the number 32 into two parts such that the quotient of the greater divided by the less will be to that of the less divided by the greater in the ratio of 25 to 9.

10. Find two numbers such that the sum of their squares will be to the difference of their squares as 17 to 8, and the difference of their squares to the difference of their cubes as 8 to 49.

11. Divide $ 121 among A, B, and C, so that A's share will be to B's as 4 to 5, and B's to C's as 9 to 8.

12. The area of a rectangular field is 3 acres, and its length to its breadth as 6 to 5. Find the dimensions.

13. The number of dollars A has is to the number B has as 9 to 5. By obtaining one half of A's money, B will have $ 2 more than A had at first. How much money has each?

14. In a square-hewn block of stone containing 5 cubic feet, the length is to the breadth as 9 to 5, and the breadth to the thickness as 5 to 3. Find the dimensions of the stone.

15. A man's age is to that of his wife as 9 to 8. Ten years ago their ages were as 13 to 11. What are their ages?

16. Of two houses, one cost $ 1000 more than the other, and the ratio of their prices was as 3 to 2. Find the cost of each.

VARIATION.

537. When two quantities are so related that one increases or diminishes in the same ratio as the other, the first is said *to vary as the second.*

Thus, if a train of cars run a miles per hour, the distance run in 2 hours will be $2\,a$ miles, in 3 hours $3\,a$ miles, and so on; that is, the distance will vary as the time varies.

538. This variation of two numbers is denoted by the sign $\propto$ written between them, called the **sign of variation,** and read "varies as." Thus, $a \propto b$ is read "a varies as b."

539. A number which in any particular problem changes its value is called a **variable,** and a number that has a fixed value is called a **constant.** Thus, in the example given above, the number of miles per hour (a) is a constant, while the numbers denoting time and distance are both variables.

540. When two variables are so related that if one be given the other can be found, one is said to be a **function** of the other. Thus, for example, if a steamer sails at a given speed, the distance sailed in a certain time will depend on the time, and, if the time be given, the distance can be found; and hence the distance is a function of the time.

541. When one number varies as another, their corresponding values have a *constant ratio.*

Thus, if a_1, a_2, a_3, etc., denote the increase of one variable, and b_1, b_2, b_3, etc., the corresponding increase of the other, so that

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}, \quad (1); \text{ and } \quad \frac{a_2}{a_3}=\frac{b_2}{b_3}, \qquad (2)$$

then, from (1), $\frac{a_1}{b_1}=\frac{a_2}{b_2}$; and from (2), $\frac{a_2}{b_2}=\frac{a_3}{b_3}$. (§ 525.)

Hence $$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}\cdots = \text{a constant ratio.}$$

Hence, if $a \propto b$, their common ratio is found by dividing a by b, or a_2 by b_2, or a_3 by b_3, and so on. If m denotes this common ratio, $\frac{a}{b}=m$, and $a=bm$.

542. There are many ways in which two variables may be related. Four of the more important cases are here presented.

543. I. When two numbers so vary that their corresponding values have a constant ratio, they are said to vary *directly.*

Thus, if $a \propto b$ directly, $\frac{a}{b}=m$, their constant ratio, and $a=bm$. Hence, if, in $\frac{a}{b}=m$ or $a=bm$, $a=12$ and $b=4$, $m=3$ and $a=3b$.

1. A workman earns \$40 a month. How much will he earn in 5 months? In 12 months?

Which number in this problem is the constant? Give the two variables. Which is the function of the other?

2. The base of a rectangle is 10 inches. What will be its area if the altitude be 1 inch? 3 inches? 8 inches?

Which number is the constant? Give the two variables. Which is the function of the other?

544. II. When one number varies as the reciprocal of another, the numbers are said to vary *inversely*. Thus, when a varies inversely as b, $a \propto \frac{1}{b}$; and then $a = \frac{1}{b} \times m$, and $ab = m$.

For example, the time required to do a given work varies inversely as the number of workmen employed. If 2 men can do the work in 6 days, 4 men can do it in 3 days; that is, *twice* as many men will do it in *one half* of the time.

3. Three men can dig a ditch in 12 days. How long will it take 6 men to dig it? 9 men? 12 men?

Which number in this problem is the constant? What numbers are the variables? Which is the function of the other?

4. If a quantity of oats will feed 4 horses 9 days, how many horses will it feed 3 days?

545. In some cases one quantity varies inversely as the *square* of another. Thus, $a \propto \frac{1}{b^2}$.

For example, the illumination of a candle decreases inversely as the square of the distance from it. If it gives a certain illumination at a distance of 1 foot, the illumination at 2 feet will be only $\frac{1}{4}$ as much; at 3 feet, only $\frac{1}{9}$ as much; and so on.

5. If the illumination of a gas jet at 25 feet is x, what will it be at 50 feet? At 100 feet?

6. If the attraction of a magnet for a piece of iron at the distance of $\frac{1}{30}$ of an inch is x, what will be the attraction at $\frac{2}{30}$ of an inch? $\frac{8}{30}$ of an inch? $\frac{5}{30}$ of an inch?

546. III. When one number varies as the product of two other numbers, it is said to vary as the two others *jointly*. Thus, if $a \propto bc$, a varies jointly as $b \times c$.

For example, the area of a rectangle varies as the product of its base and altitude.

7. If the area of a rectangle with a given base and altitude is x, what will be its area when the base and altitude are each doubled? When the base is trebled and the altitude doubled?

The pressure of gas varies directly as its density, and also as its temperature; and hence, if the pressure of gas at a given density and temperature is represented by x, its pressure when its density is doubled, and its temperature increased one half, will be expressed by $x \times 2 \times \frac{3}{2} = 3x$. Hence,

8. If the pressure of gas at a given density and temperature is 15 pounds to the square inch, what will be its pressure if the density be trebled, and the temperature reduced one half?

547. IV. A number is said to vary *directly as a second number, and inversely as a third*, when it varies as the product of the second and the reciprocal of the third.

Thus, a varies directly as b, and inversely as c, when $a \propto b \times \frac{1}{c}$; that is, when $a : b \times \frac{1}{c}$ is constant.

When a number varies as the quotient of two numbers, it varies directly as the dividend, and inversely as the divisor. Thus, if $a \propto \frac{b}{c}$, a varies directly as b, and inversely as c.

9. The volume of gas varies as the absolute temperature, and inversely as the pressure. If the volume is represented by x when the pressure is 15 and the temperature 300, what will be the volume when the pressure is 20 and the temperature 350?

548. In all four of the cases of variation presented above, the *constant* can be determined when any one set of corresponding values is given.

Thus (1), if $a \propto b$, $\frac{a}{b} = m$; (2), if $a \propto \frac{1}{b}$, $ab = m$; (3), if $a \propto bc$, $\frac{a}{bc} = m$; and (4), if $a \propto b \times \frac{1}{c}$, $a = \frac{b}{c} \times m$. $\therefore \frac{ac}{b} = m$.

549. If a depends only on b and c, and $a \propto b$ when c is constant, and $a \propto c$ when b is constant, then, when both b and c vary, $a \propto bc$.

Let a, b, c; a', b', c; a'', b', c',—be three sets of corresponding values.

Then, since c is in the first and second, $\frac{a}{a'} = \frac{b}{b'}$; (1)

also, since b' is in the second and third, $\frac{a'}{a''} = \frac{c}{c'}$. (2)

Multiplying (1) by (2), $\frac{a}{a'} \times \frac{a'}{a''} = \frac{bc}{b'c'}$;

whence $\frac{a}{a''} = \frac{bc}{b'c'}$. $\therefore \frac{a}{bc} = \frac{a''}{b'c'}$.

Hence a varies as bc.

For example, the area of a rectangle varies as its altitude when its base is constant, and as its base when its altitude is constant, and as the product of its base and altitude when both vary.

550. The simplest method of solving problems in variation is to convert the variations into equations.

For example, if $a \propto b$ and $b \propto c$, show that $a \propto c$.

By § 543 we have $a = bm$ (1), and $b = cn$ (2), m and n being constant ratios.

Multiplying (1) by (2), $ab = bcmn$;

whence $a = cmn$. $\therefore a \propto c$.

10. If $x \propto y$, and $x = 5$ when $y = 3$, find x when $y = 9$.

11. If $a \propto b$, and b is 9 when a is 6, what is b when a is 9?

12. $a \propto b$ and $b \propto c$: show that $ac \propto b^2$.

13. $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$: show that $a \propto c$.

14. $x \propto yz$: if $x = 2$ when $y = 4$ and $z = 3$, what will x equal when $y = 2$ and $z = 9$?

15. If the area of a rectangle is x when its base is a and its altitude b, what will be its area when its base is $3a$ and its altitude $\frac{2}{3}b$?

16. The volume of a sphere varies as the cube of its diameter. If the volume of a sphere 2 inches in diameter is 4.188 cu. in., what is the volume of a sphere 5 inches in diameter?

17. The area of a sphere varies as the square of its diameter, and the surface of a sphere 5 inches in diameter is 78.54 sq. in. What is the surface of a sphere 10 inches in diameter?

18. If the volume of a sphere varies as the cube of its diameter, how many spheres 3 inches in diameter equal a sphere 12 inches in diameter?

19. The area of a circle varies as the square of its diameter. How many circles 4 inches in diameter equal one 20 inches in diameter?

20. The velocity of a falling body varies as the time during which it has fallen from rest. If the velocity of a falling ball at the end of 2 seconds is 64 feet, what will be its velocity at the end of 6 seconds?

21. The distance a body falls from rest varies as the square of the time it falls. If a ball falls 144 feet in 3 seconds, how far will it fall in 12 seconds?

22. The quantity of water that flows through a circular pipe varies as the square of the pipe's diameter. If 10 gallons a minute flow through an inch pipe, how many gallons per minute will flow through a 4-inch pipe?

CHAPTER XIX.

PROGRESSIONS.

551. A **series** is a succession of numbers formed according to some fixed law, called the *law of the series.* The successive numbers are called the **terms** of the series.

552. A series that consists of a limited number of terms is called a **finite series,** and one that consists of an unlimited number of terms is called an **infinite series.** If a finite series be considered as continued indefinitely in either or both directions, it becomes an infinite series.

ARITHMETICAL PROGRESSION.

553. An **arithmetical progression** is a series in which each term is obtained by adding a constant number to the preceding term. The constant number added is called the **common difference.** An arithmetical progression is denoted by A. P.

554. If the common difference is *positive,* the series is said to be *increasing.* Thus the series $3, 5, 7, 9, 11, \cdots$ is an increasing arithmetical progression, in which the common difference is 2.

555. If the common difference is *negative,* the series is said to be *decreasing.* Thus the series $12, 9, 6, 3, 0, -3, -6, \cdots$, is a decreasing arithmetical progression, in which the common difference is -3.

556. The first and last terms of a finite arithmetical series are called the **extremes;** and the intermediate terms, the

arithmetical means. Thus, in the series 2, 6, 10, 14, 18, with five terms, 2 and 18 are the extremes; and 6, 10, and 14, the arithmetical means.

557. If a denotes the first term of an arithmetical progression, and d the common difference, then $a + d$ will denote the second term, $a + 2d$ the third term, $a + 3d$ the fourth term; and so on to the nth term, which will be denoted by $a+(n-1)d$.

If the series consists of n terms, and the nth or last term be denoted by l, we have the equation

$$l = a + (n - 1)d. \qquad (A)$$

558. This formula contains four symbols, denoting as many different numbers. If any three of these numbers are given, the fourth may be found *by substituting the given numbers for their symbols in* (A), *and solving the resulting equations.*

1. Find the thirteenth term of the A. P. 6, 10, 14, 18, ...

Here $n = 13$, $a = 6$, and $d = 10 - 6 = 4$.

Substituting 13 for n, 6 for a, and 4 for d, in (A), we obtain

$$l = 6 + (13 - 1)\,4 = 6 + 12 \times 4 = 54.$$

Hence the thirteenth or last term is 54.

The common difference in a given arithmetical series may evidently be found by subtracting any term from the term which next follows it.

2. In an A. P. of 30 terms, the last term is 8, and the common difference is 2. Find the first term.

Substituting 30 for n, 8 for l, and 2 for d, in (A), we obtain

$$8 = a + (30 - 1) \times 2.$$

$$\therefore\ a = 8 - 58 = -50.$$

3. Find the number of terms of an A. P. whose first and last terms are 5 and 65 respectively, and common difference 3.

Substituting 5, 65, and 3 for a, l, and d respectively, in (A), we obtain

$$65 = 5 + (n - 1)\,3;$$

whence $$n - 1 = 20. \quad \therefore\ n = 21.$$

4. Find the common difference when the first and last terms of an A. P. of 64 terms are 9 and 16 respectively.

Substituting 64, 9, and 16 for n, a, and l respectively, in (A), we obtain

$$16 = 9 + (64 - 1)\,d.$$

$$\therefore\ d = \frac{16 - 9}{63} = \frac{1}{9}.$$

5. Insert 10 arithmetical means between 2 and 35.

The first step is to find the common difference, and hence the problem is similar to the above. Since there are 10 mean terms, the whole number of terms is 12.

Substituting 12, 2, and 35 for n, a, and l respectively, in (A), we obtain

$$35 = 2 + (12 - 1)d.$$

$$\therefore\ d = \tfrac{33}{11} = 3.$$

Hence the required series is 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32. 35.

6. The sixth term of an A. P. is 2, and the thirteenth term is 23. What is the first term?

Make the given sixth term the first term of a series of 8 terms in which $a = 2$, $l = 23$, and $n = 8$.

Substituting 2, 23, and 8 for a, l, and n respectively, in (A), we obtain

$$23 = 2 + (8 - 1)d = 2 + 7d.$$

$$\therefore\ d = 3.$$

Now consider the sixth term (2) the last term of a series of 6 terms, and we have

$$2 = a + 5 \times 3.$$

$$\therefore\ a = -13,\ \text{the first term of the series.}$$

7. Find the fifteenth term of the A. P. 17, 14, 11, ….

8. Find the sixteenth term of the A. P. $-12, -7, -2, \ldots$.

9. Find the fifteenth term of the A. P. $7\frac{2}{5}$, $6\frac{4}{5}$, $6\frac{1}{5}$, ….

10. Given $n = 15$, $d = 4$, $l = 70$: find a.

11. Given $n = 50$, $d = \frac{2}{3}$, $l = 5\frac{4}{7}$: find a.

12. Given $n = 105$, $d = .07$, $l = -2.35$: find a.

13. Given $d = 3$, $a = 12$, $l = 72$: find n.

14. Given $d = -\frac{1}{5}$, $a = -5\frac{1}{3}$, $l = -101\frac{14}{15}$: find n.

15. Given $d = .5871$, $a = .29$, $l = 59$: find n.

16. Given $n = 22$, $a = 4$, $l = 130$: find d.

17. Given $n = 58$, $a = .33$, $l = 75$: find d.

18. Insert 7 arithmetical means between 5 and 29.

19. Insert 8 arithmetical means between 1 and -5.

20. Insert 9 arithmetical means between -6 and 0.

559. A general formula for finding the *sum* of n terms of an arithmetical progression may be obtained as follows:

Let s denote the sum of the terms, and l the last term; then

$$s = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + l, \qquad (1)$$

also
$$s = l + (l - d) + (l - 2d) + (l - 3d) + \cdots + a, \qquad (2)$$

the sum in (2) being written in reverse order.

Adding (2) to (1), term to term, we obtain

$$2s = (a+l)+(a+l)+(a+l)+(a+l)+\cdots+(a+l) \text{ to } n \text{ terms.}$$

Hence
$$2s = n(a + l),$$

and
$$s = \frac{n}{2}(a + l). \qquad (B)$$

560. This formula also contains four symbols, and may be employed to obtain solutions for as many classes of problems.

21. Find the sum of 15 terms of an A. P. that has for its extremes 7 and 45.

Here $n = 15$, $a = 7$, $l = 45$; and substituting these values for n, a, and l respectively, in (B), we obtain

$$s = \frac{15}{2}(7 + 45) = 15 \times 26 = 390.$$

22. The sum of 40 terms of an A. P. is -560, and the last term is -36. Find the first term.

Substituting 40, -560, and -36, for n, s, and l, in (B), we obtain

$$-560 = \frac{40}{2}(a - 36),$$

whence
$$a = \frac{720 - 560}{20} = 8.$$

The last term of an A. P. is found in like manner when s, a, and n are given.

23. The sum of the terms of a series is 272, and the extremes are -13 and 45. Find the number of terms.

Substituting -13, 45, and 272, for a, l, and s respectively, in (B), we obtain

$$272 = \frac{n}{2}(-13 + 45),$$

whence
$$n = \frac{272}{16} = 17.$$

24. Given $a = 11$, $l = 25$, $n = 14$: find s.

25. Given $a = -13$, $l = 105$, $n = 63$: find s.

26. Given $a = -3.7$, $l = -1.3$, $n = 42$: find s.

27. Given $s = 598$, $l = 39$, $n = 26$: find a.

28. Given $s = 1295$, $l = 80$, $n = 37$: find a.

29. Given $s = 1184$, $a = 11$, $n = 32$: find l.

30. Given $s = 116.375$, $a = -3.5$, $n = 19$: find l.

31. Given $s = 336$, $a = 3$, $l = 25$: find n.

32. Given $s = 30.225$, $a = .75$, $l = 1.2$: find n.

33. Given $s = 147\sqrt{5}$, $a = 3\sqrt{5}$, $l = 18\sqrt{5}$: find n.

561. The fundamental formulas (A) and (B) contain, in all, five symbols; and such is the relation between the numbers represented, that, if any three of them are given, the other two can be found.

562. Of the twenty classes of problems thus arising, eight can be solved by a single formula, as above exemplified; but the remaining twelve classes require both formulas, since they

each involve the numbers denoted by d and s, given or required, and d occurs only in (A), and s only in (B).

34. Find the sum of 30 terms of the series 7, 10, 13, ...

Here $n = 30$, $a = 7$, and $d = 3$, and s is required.

Substituting their values for n, a, and d, in (A) and (B), we obtain

$$l = 7 + (30 - 1) \times 3 = 94, \quad (1)$$

$$s = \tfrac{30}{2}(7 + l) = 105 + 15\,l. \quad (2)$$

Substituting in (2) the value of l in (1), we have

$$s = 105 + 15 \times 94 = 1515.$$

35. How many successive terms of the series 11, 9, 7, ..., must be taken, that their sum may be 27?

Here $a = 11$, $d = -2$, and $s = 27$, and n is required.

Substituting their values for a, d, and s, in (A) and (B), we obtain

$$l = 11 + (n - 1) \times (-2) = 13 - 2\,n, \quad (1)$$

$$27 = \frac{n}{2}(11 + l). \quad \therefore\ 54 = 11\,n + ln. \quad (2)$$

Substituting in (2) the value of l found from (1), simplifying, etc., we obtain $n^2 - 12\,n = -27$;

solving the equation, $n = 9$ or 3.

For $n = 9$, the A. P. is 11, 9, 7, 5, 3, 1, -1, -3, -5, the sum being 27.

For $n = 3$, the A. P. is 11, 9, 7, the sum being 27.

To find the value of l, substitute both values of n in (1), obtaining, for $n = 9$, $l = 13 - 2 \times 9 = -5$; for $n = 3$, $l = 13 - 2 \times 3 = 7$.

PROBLEMS.

Find the sum of

1. 30 terms of the A. P. 5, 7, 9, ...
2. 16 terms of the A. P. 9, 5, 1, ...
3. 36 terms of the A. P. $1\frac{1}{2}$, 2, $2\frac{1}{2}$, ...
4. 15 terms of the A. P. $-2\frac{1}{8}$, $-1\frac{2}{8}$, -1, ...
5. 19 terms of the A. P. 5.3, 6.1, 6.9, ...
6. 100 terms of the A. P. 1, 2, 3, ...
7. n terms of the A. P. 1, 2, 3, ...

8. Find the sum of n terms of the A. P. 1, 3, 5, ….

9. Find the nth term of each series in 7 and 8.

10. The extremes of an A. P. are 2 and 30, and the common difference is 2. Find the sum of the series.

11. The first term of an A. P. is 10, the number of terms 10, and the sum of the terms 85. Find the common difference.

12. The common difference of an A. P. is 2, the last term 23, and the number of terms 11. Find the sum of the series.

13. The common difference of an A. P. is 2, the number of terms 12, and the sum of the terms 90. Find the last term.

14. The number of terms of an A. P. is 13, the common difference -7, and the sum of the terms 39. Find the first term.

15. The last term of an A. P. is 52, the common difference 5, and the sum of the series 297. Find the number of terms.

16. The fourth term of an A. P. is -1, and the tenth term is 3. What is the first term? What is the sum of the series?

17. A man paid \$3000 in 6 installments. The first was \$400, and the last \$600, all the payments being in an A. P. What was the amount of each intermediate payment?

18. The cost of sinking a well was \$45, \$1 being paid for sinking the first yard of depth, \$1.50 for the second, \$2 for the third, and so on. What was the depth of the well?

19. A man began saving by putting by 1 cent on New Year's Day, 2 cents on the next day, 3 cents on the next, and so on. In how many days would he have put by \$98.70?

20. A load of 100 fence posts was laid down at the corner of a field. If, leaving 1 post at the corner, a man should carry the rest one by one, to lay them down in a straight line at intervals of 3 yards, how far would he walk to accomplish his task?

GEOMETRICAL PROGRESSION.

563. A **geometrical progression** is a series in which each term bears a constant ratio to the preceding one. A geometrical progression is denoted by G. P.

564. This constant or common ratio is called the **ratio of the series**; and the series is *increasing* or *decreasing* according as its ratio is numerically greater or less than unity.

Thus, the series 2, 6, 18, ···, is an increasing geometrical progression in which the ratio is 3; and 64, 16, 4, ···, is a decreasing progression in which the ratio is $\frac{1}{4}$.

565. If the ratio be positive, all the terms will be positive; but if the ratio be negative, the terms will be alternately positive and negative, as in the series $5, -10, +20, -40, \cdots$, in which the ratio is -2.

A geometrical progression is a continued proportion, as defined in § 520.

566. When a geometrical progression consists of a definite number of terms, the first and last are called **extremes**; and the intermediate terms are the **geometrical means.**

A geometrical series may be considered as extending indefinitely in either direction, and it is then called an *infinite geometrical series.*

567. If a denotes the first term of a G. P., and r the ratio, then ar will denote the second term, ar^2 the third term, ar^3 the fourth term; and so on to the nth term, which will be denoted by ar^{n-1}, it being the $(n-1)$th term after the first.

Hence the last term (l) in a series of n terms will be designated by the formula

$$l = ar^{n-1}. \qquad (A)$$

568. This formula, like the corresponding one in arithmetical progression, contains four symbols, denoting as many

numbers; and, if any three of these numbers are given, the fourth can be found *by substituting the given numbers for their symbols in* (A), *and solving the resulting equation.*

1. Find the ninth term of the G. P. 243, 81, 27, ...

Here $a = 243$, $n = 9$, and $r = \frac{81}{243} = \frac{1}{3}$.

Substituting their values for a, n, r, in (A), we obtain

$$l = 243 \times \left(\frac{1}{3}\right)^8 = 243 \times \frac{1}{243 \times 27} = \frac{1}{27}.$$

It is evident that the ratio in a given G. P. is found by dividing any given term by the one preceding it.

2. Find the first term of the G. P. whose eleventh term is 3072, and whose ratio is -2.

Substituting their values for n, l, r, in (A), we obtain

$$3072 = a \times (-2)^{10} = a \times 1024;$$

whence $$a = 3072 \div 1024 = 3.$$

3. Find the ratio of the G. P. whose first and eighth terms are 1715 and $\frac{5}{2401}$ respectively.

Substituting their values for a, l, n, in (A), we obtain

$$\frac{5}{2401} = 1715\, r^7;$$

whence $$r^7 = \frac{5}{2401 \times 1715} = \frac{1}{2401 \times 343}.$$

$$\therefore\ r = \sqrt[7]{\frac{1}{2401 \times 343}} = \frac{1}{7}.$$

4. Insert 4 geometrical means between $-\frac{1}{54}$ and 144.

The first step is to find the ratio, as in the preceding problem. The number of terms is $4 + 2$, or 6.

Substituting their values for a, l, n, in (A), we obtain

$$144 = -\tfrac{1}{54} \times r^5,$$

whence $$r = -6.$$

Hence the required means are

$$\tfrac{6}{54},\ -\tfrac{36}{54},\ \tfrac{216}{54},\ -\tfrac{1296}{54};\ \text{or}\ \tfrac{1}{9},\ -\tfrac{2}{3},\ 4,\ -24.$$

5. Find the eighth term of the G. P. 1, 2, 4, ...

6. Find the tenth term of the G. P. 28, 7, $\frac{7}{4}$, ...

7. Find the seventh term of the G. P. $1, -\frac{1}{5}, \frac{1}{25}, \cdots$.

8. Given $l = 1458$, $n = 6$, $r = 3$: find a.

9. Given $l = 960$, $n = 7$, $r = -2$: find a.

10. Given $l = -567$, $n = 5$, $a = -7$: find r.

11. Given $l = \frac{128}{729}$, $n = 7$, $a = 2$: find r.

12. Insert 3 geometrical means between 2 and $\frac{32}{81}$.

13. Insert 7 geometrical means between $\frac{1}{3}$ and 9.

14. Insert 5 geometrical means between $\frac{1}{5}$ and $\frac{64}{1715}$.

15. Show that the geometrical mean between two numbers is the square root of their product.

569. When a, l, and r are given in order to find n, the process leads to an *exponential equation;* that is, an equation in which the unknown number occurs as an *exponent.* The solution of this form of an equation can usually be effected only by the use of logarithms, as shown in § 613.

570. A general formula for finding the *sum* of n terms of a G. P. may be obtained as follows:

Let s denote the sum of the terms;

then $$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \quad (1)$$

Multiplying by r, $$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (1) from (2),

$$rs - s = ar^n - a, \text{ or } s(r-1) = a(r^n - 1);$$

whence $$s = a \times \frac{r^n - 1}{r - 1}. \quad (B)$$

571. This formula also contains four symbols, and may be employed for the solution of as many classes of problems; but in two cases it may lead to equations of a higher degree than the second.

16. Find the sum of 8 terms of the G. P. 2, 6, 18 ···.

Here $a = 2,\ n = 8,\ r = 3.$

Substituting their values for a, n, r, in (B), we obtain

$$s = 2 \times \frac{3^8 - 1}{3 - 1} = 3^8 - 1 = 6560.$$

17. The sum of 10 terms of a G. P. is -1705, and the ratio is -2. Find the first term.

Substituting their values for n, r, s, in (B), we obtain

$$-1705 = a \times \frac{(-2)^{10} - 1}{-2 - 1} = a \times -341;$$

whence

$$a = \frac{-1705}{-341} = 5.$$

18. Find the sum of 6 terms of a G. P. the first and last terms of which are 3 and $\frac{3}{32}$ respectively.

Substituting their values for n, a, l, in (A), we obtain

$$r = \tfrac{1}{2}.$$

Substituting their values for a, n, r, in (B), we obtain

$$s = 3 \times \tfrac{126}{64} = 5\tfrac{29}{32}.$$

572. When $r < 1$ numerically, the formula (B) is more convenient in the form

$$s = a \times \frac{1 - r^n}{1 - r}.$$

When $r = 1$, $s = a(1 + r + r^2 + r^3 + \cdots + r^{n-1})$, and hence $s = an$.

Infinite Series.

573. If $r < 1$, the value of r^n decreases as n increases; and hence, as n is indefinitely increased, r^n is indefinitely diminished, and *thus approaches zero for its limit.* Therefore the above formula becomes

$$s = \frac{a}{1 - r}, \qquad (B')$$

$\frac{a}{1 - r}$ being the limit towards which s approaches when $r < 1$, and both a and r are positive, and n is indefinitely increased.

This limit is properly called the limit of the sum of a

converging geometrical series as *n approaches infinity*, but for convenience it is usually called *the limit of the series to infinity*.

An infinite number is denoted by the symbol ∞.

For converging and diverging series, see Chapter XXI., p. 336, note.

19. Find the limit of the sum of $1 + \frac{1}{2} + \frac{1}{4} \cdots$ to infinity.

Here $a = 1$, $r = \frac{1}{2}$; and hence, by (B'), $s = \dfrac{1}{1 - \frac{1}{2}} = 2$.

This may be illustrated by taking a slip of paper, say 2 inches long, dividing it into two equal parts, taking away one of these parts, and bisecting the other; and so on indefinitely. It is evident that the sum of the parts taken away can never exceed 2 inches.

20. Find the sum of 10 terms of the G. P. $2 - 6 + 18 - \cdots$.

21. Find the sum of 7 terms of the G. P. $75 + 15 + 3 + \cdots$.

22. Find the sum of 8 terms of the G. P. $1 - \frac{2}{3} + \frac{4}{9} - \cdots$.

23. Given $n = 8$, $r = 2$, $s = 765$: find a.

24. Given $n = 7$, $r = -3$, $s = 547$: find a.

25. Given $n = 10$, $r = \frac{1}{2}$, $s = 15\frac{63}{64}$: find a.

26. Given $n = 7$, $r = .5$, $s = 1943.1$: find a.

Find the limit of the sum, to infinity, of

27. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$.

28. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

29. $\frac{2}{5} + \frac{2}{5} + \frac{4}{15} + \cdots$.

30. $3 + \frac{3}{5} + \frac{3}{25} + \cdots$.

31. $1 + \dfrac{1}{n} + \dfrac{1}{n^2} + \cdots$.

32. Find the value of the recurring decimal $.53434 + \cdots$.

The repeating part, $.03434 +$, is an infinite geometrical series, the first term being $\frac{34}{1000}$, the second $\frac{34}{1000000}$, and so on.

Substituting $\frac{34}{1000}$ for a, and $\frac{1}{100}$ for r, in (B'), we obtain

$$s = \tfrac{34}{1000} \div (1 - \tfrac{1}{100}) = \tfrac{34}{990}.$$

Hence the value of the decimal is $\frac{5}{10} + \frac{34}{990} = \frac{529}{990}$.

A recurring decimal is also called a *circulating* or a *repeating* decimal. The repeating part, called the *repetend*, is denoted by a dot placed over the first and the last of its figures; thus, $.5\dot{3}\dot{4}$.

Find the value of

33. $.2\dot{4}\dot{1}$. 35. $.3\dot{5}\dot{4}$. 37. $8.\dot{9}0\dot{1}$.

34. $6.\dot{5}\dot{3}$. 36. $.45\dot{1}2\dot{4}$. 38. $54.\dot{3}2\dot{1}$. (*See also* § 688.)

574. Of the twenty possible cases in geometrical progression, eight can be solved by either (*A*) or (*B*), as shown above. The twelve remaining cases — that is, all those cases in which both the numbers denoted by l and s are involved — require the use of both formulas.

39. The extreme terms of a G. P. are 3 and 192, and the ratio 2. Find the sum of the terms.

Substituting their values for a, l, and r, in (*A*) and (*B*), we obtain

$$192 = 3 \times 2^{n-1}. \quad \therefore\ 2^{n-1} = 64, \text{ or } 2^n = 128. \tag{1}$$

$$s = 3 \times \frac{2^n - 1}{2 - 1}. \quad \therefore \text{ substituting 128 for } 2^n,\ s = 381. \tag{2}$$

40. Find the seventh term of the G. P. whose ratio is $\frac{3}{2}$, and the sum of the series $42\frac{3}{4}\frac{3}{8}$.

Substituting their values for n, r, s, in (*A*) and (*B*), we obtain

$$l = a \times (\tfrac{3}{2})^6 = a \times \tfrac{7}{6}\tfrac{2}{4}\tfrac{9}{}, \tag{1}$$

$$\frac{2059}{48} = a \times \frac{\frac{2187}{128} - 1}{\frac{3}{2} - 1} = a \times \frac{2059}{64}. \quad \therefore\ a = \tfrac{4}{3}. \tag{2}$$

Substituting $\frac{4}{3}$ for a in (1),

$$l = \tfrac{4}{3} \times \tfrac{7}{6}\tfrac{2}{4}\tfrac{9}{} = \tfrac{2}{1}\tfrac{4}{6}\tfrac{3}{}.$$

575. When the number denoted by n is required, the process leads to an exponential equation (§ 569); and in some cases, if $n > 2$, to equations of higher degree than the second. The foregoing examples may sufficiently indicate how to proceed in soluble cases. Sometimes the value of n in an exponential equation is readily found by inspection, as in $2^n = 128$, occurring in the solution of Example 39 above.

PROBLEMS.

1. Given $a=4$, $l=2916$, $n=7$: find s and r.

2. Given $a=5$, $l=320$, $n=7$: find s and r.

3. Given $r=4$, $l=1024$, $n=9$: find s and a.

4. Given $r=2$, $l=20480$, $n=14$: find a and s.

5. Given $r=3$, $s=2391484\frac{1}{3}$, $a=\frac{1}{3}$: find l.

6. Given $l=65536$, $s=74898\frac{9}{512}$, $a=\frac{1}{512}$: find r and n.

7. What is the sum of the series $2, \frac{2}{3}, \frac{2}{9}, \cdots$, to infinity?

8. What is the sum of $1, -\frac{1}{x^2}, \frac{1}{x^4}, -\frac{1}{x^6}, \cdots$, to infinity?

9. The extremes of a G. P. are 2 and $\frac{2}{2187}$, and the ratio is $\frac{1}{3}$. What is the sum of the series?

10. The sum of the third and fourth terms of a G. P. is 18, and the difference of the third and fifth terms is -36. Find the series.

11. A population increases yearly in geometrical progression, and in 4 years is raised from 10,000 to 14,641. Find the ratio of annual increase.

12. There are 3 numbers in geometrical progression whose sum is 62; and the sum of the first and second is to the sum of the second and third as 1 to 5. Find the numbers.

13. The difference between the first and second of 4 numbers in geometrical progression is 15, and the difference between the third and fourth is 540. Find the numbers.

14. A person wishes to send \$ 9950, besides as much more as will cover the express charges for the whole sum at the rate of $\frac{1}{2}\%$. How much should he send in all?

15. Some grains of wheat found in the crop of a wild fowl being planted, one germinated and produced 50 sound grains. These again being sown produced a crop of 2500 grains. Of how many grains would the crop of the sixth year consist, supposing the grain to increase every year at the same rate?

16. An elastic ball falls from a sufficient height to rebound 30 feet, and at each successive rebound rises $\frac{1}{3}$ of the distance of the previous one. How many feet will the ball pass over in 5 rebounds? How many feet before it comes to rest?

17. There are 4 numbers in geometrical progression, and the first is 21 less than the fourth, and the difference of the extremes divided by the difference of the means is $3\frac{1}{2}$. Find the numbers.

HARMONIC PROGRESSION.

576. Three numbers are said to be in *harmonic proportion* when the first is to the third as the difference of the first and second is to the difference of the second and third.

Thus, if a, b, c, are in harmonic proportion,

$$a : c = a - b : b - c, \text{ or } a : c = b - a : c - b.$$

577. An **harmonic progression** is a series in which the first of *any* three consecutive terms is to the third as the difference of the first and second is to the difference of the second and third. An harmonic progression is denoted by H. P.

Thus, if a, b, c, d, e, are in harmonic progression,

$$a : c = a - b : b - c; \quad b : d = b - c : c - d; \text{ and } c : e = c - d : d - e.$$

578. If a, b, c, are in harmonic progression, we have, by definition,

$$a : c = a - b : b - c;$$

whence $$c(a - b) = a(b - c).$$

Dividing by abc, $$\frac{a - b}{ab} = \frac{b - c}{bc};$$

whence $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Hence, *if given numbers are in harmonic progression, their reciprocals are in arithmetical progression.*

579. Problems relating to harmonic series are usually best solved *by writing the reciprocals of the terms as an arithmetical progression, and then solving the resulting equations.*

1. The second term of an H. P. is 2, and the fourth term 6. Find the series.

The second and fourth terms of the corresponding A. P. are $\frac{1}{2}$ and $\frac{1}{6}$.

Let a be the first term, and d the common difference. Then

$$a + d = \tfrac{1}{2},$$
$$a + 3d = \tfrac{1}{6};$$

whence $a = \frac{2}{3}$, and $d = -\frac{1}{6}$.

The A. P. is $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, etc.;

hence the H. P. is $\frac{3}{2}, 2, 3, 6$, etc.

2. Insert 3 harmonic means between 3 and 15.

The first and fifth terms of the corresponding A. P. are $\frac{1}{3}$ and $\frac{1}{15}$; and hence $d = (\frac{1}{15} - \frac{1}{3}) \div 4 = -\frac{1}{15}$.

Thus the A. P. is $\frac{1}{3}, \frac{4}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}$.

Hence the 3 harmonic means are $\frac{15}{4}$, 5, $\frac{15}{2}$.

3. The first term of an H. P. is 2, and the third term is 6. What is the second term?

4. The second term of an H. P. is b, and the third term is c. What is the first term?

5. The first term of an H. P. is 2, and the fourth term is 6. Find the mean terms.

6. Insert 5 harmonic means between $\frac{1}{3}$ and $\frac{1}{18}$.

7. Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{2}{7}$.

8. Insert 4 harmonic means between $\frac{1}{2}$ and $\frac{1}{12}$.

9. The first term of an H. P. is 1, and the third term $\frac{1}{3}$. Find the sixth term.

10. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, are in arithmetical progression. Show that $a - b : b - c = a : c$.

CHAPTER XX.

LOGARITHMS.

580. The **logarithm** of a number is the exponent of the power to which a fixed number, called the **base**, must be raised in order to produce the given number.

Thus, since $3^4 = 81$, 4 is the logarithm of 81 to the base 3; since $8^3 = 512$, 3 is the logarithm of 512 to the base 8; and, generally, if $a^x = m$, x is the logarithm of m to the base a.

581. The base of the system of logarithms in common use is 10, the basis of the decimal notation; and the system is called the **common system.**

582. Since $10^0 = 1$, $10^{-1} = \frac{1}{10} = .1$,

$10^1 = 10$, $10^{-2} = \frac{1}{10^2} = .01$,

$10^2 = 100$, $10^{-3} = \frac{1}{10^3} = .001$,

$10^3 = 1000$, $10^{-4} = \frac{1}{10^4} = .0001$,

$10^4 = 10000$, and so on,

the numbers 0, 1, 2, 3, ···, are the logarithms of the successive *positive* integral powers of 10; and -1, -2, -3, -4, ···, are the logarithms of the successive *negative* integral powers of 10.

Thus (*log* being an abbreviation for logarithm),

$\log 1 = 0$, $\log .1 = -1$,

$\log 10 = 1$, $\log .01 = -2$,

$\log 100 = 2$, $\log .001 = -3$, and so on.

583. The logarithms of numbers between the integral powers of 10 are evidently fractional. Thus, the logarithms of the numbers between

100 and 1000 are $2 +$ a decimal,
10 and 100 are $1 +$ a decimal,
1 and 10 are $0 +$ a decimal,
1 and .1 are $-1 +$ a decimal,
.1 and .01 are $-2 +$ a decimal,
.01 and .001 are $-3 +$ a decimal,
and so on.

Thus, with — above negative characteristics, the logarithm of

5 is 0.69897, .5 is $\bar{1}$.69897,
25 is 1.39794, .025 is $\bar{2}$.39794,
225 is 2.35218, .00225 is $\bar{3}$.35218.

584. It is evident from the above that a logarithm consists of two parts: (1) a positive or negative integral number, called the **characteristic**; and (2) a positive fractional part, called the **mantissa.** Thus, in the logarithm 2.78176 (log 605), the 2 is the characteristic, and the decimal .78176 is the mantissa.

The characteristic is so called because it shows, as will be explained later, the number of orders, integral or decimal, in the corresponding number.

585. It is also evident that the characteristic of the logarithm of a number greater than 1 is *positive*, and that the characteristic of the logarithm of a number less than 1 and greater than 0 is *negative.* The mantissas of all logarithms are positive.

586. There are three ways of writing a logarithm when its characteristic is negative, as follows:

(1) $\bar{2}$.41162; (2) .41162 − 2; (3) 8.41162 − 10.
$\bar{12}$.33244; .33244 − 12; 8.33244 − 20.

The first method is simple, and is used herein. The sign — is written *over* the characteristic to show that it alone is negative, the mantissa being always positive.

587. The number that corresponds to a given logarithm is called the **antilogarithm,** which is abbreviated as *antilog.* Thus, 605 is the antilog of the logarithm 2.78176.

588. The logarithms of a series of numbers arranged in tabular form is called a *table of logarithms* (§ 611).

The logarithms of numbers, as now found arranged in tables, were originally calculated by very laborious methods, consisting essentially in repeated extractions of the square root. These methods now possess only an historic interest, since methods have been devised by which logarithms can be calculated with great facility. These methods, however, depend upon principles the proof of which belongs to a more advanced treatise on algebra.

PRINCIPLES.

589. The use of logarithms as a means of facilitating certain numerical computations involves principles which we now proceed to establish; and since these principles are the same, whatever may be the base of the system, let us denote the base by the general symbol a.

590. Let m and n denote any two numbers, and x and y their respective logarithms to the base a; then, by definition,

$$a^x = m. \tag{1}$$

$$a^y = n. \tag{2}$$

Multiplying together (1) and (2) member by member, we have

$$a^{x+y} = mn;$$

whence $$x + y = \log mn, \text{ by definition.}$$

Hence *the logarithm of a product is equal to the sum of the logarithms of the factors.*

It follows that the logarithm of a composite number is *the sum of the logarithms of its factors.*

591. Dividing (1) by (2) member by member, we have

$$a^{x-y} = \frac{m}{n};$$

whence $$x - y = \log\frac{m}{n}.$$

Hence *the logarithm of a quotient is equal to the difference of the logarithms of dividend and divisor.*

It follows that the logarithm of a common fraction is *the logarithm of its numerator minus the logarithm of its denominator.*

The sums and differences of numbers cannot be found by logarithms.

592. Raising both members of (1) to the power denoted by p, we have

$$a^{px} = m^p;$$

whence $$px = \log m^p.$$

Hence *the logarithm of a power is equal to the logarithm of the base of that power multiplied by the exponent of the power.*

593. Extracting the rth root of both members of (1), we have

$$a^{\frac{x}{r}} = \sqrt[r]{m};$$

whence $$\frac{x}{r} = \log\sqrt[r]{m}.$$

Hence *the logarithm of a root is equal to the logarithm of the power divided by the index of the root.*

594. If in (1) we make $m = 1$, the corresponding value of x will be zero, since $a^0 = 1$, whatever a may be (§ 123).

Hence, *in all systems of logarithms, log 1 = 0.*

595. If, again, in (1) we make $x = 1$, then, since $a^1 = a$, or $\log a = 1$, it follows that *in all systems of logarithms the logarithm of the base is unity.*

596. The application of the foregoing principles to numerical computations by logarithms involves (1) the obtaining of the logarithms of numbers from the tables, and (2), when a logarithm is given, the finding of the corresponding number or antilog. These processes will now be explained.

ARRANGEMENT AND USE OF TABLES.

597. The simplest method of presenting a table of logarithms would be to arrange the numbers in their natural order in vertical columns, and place opposite each number its logarithm; but tables thus arranged would be altogether too voluminous for use. Various expedients, therefore, are employed in order to save space in tables, and also time in their use. The great advantage, indeed, of adopting 10 as the base of the common system arises from the fact that tables of logarithms to that base can be presented in a very compact form.

598. It has already been shown that the logarithm of any number between two integral powers of 10 lies between the integers which are the logarithms of those powers (§ 583). Thus, the logarithm of the number 3265, which is between 10^3 and 10^4, must lie between 3 and 4, the exponents of the given powers of 10. This logarithm is, in fact, 3.51388.

It is thus seen that the *positive* characteristic of a logarithm is always *one less than the number of integral orders in its antilog.*

599. The mantissas of a series of logarithms increase as their antilogs increase from one power of 10 to another. This increase of mantissas is shown below for the antilogs from 1 to 10^1 and from 10^1 to 10^2, only three figures of each mantissa being given.

Antilog	1	2	3	4	5	6	7	8	9	10
	10	20	30	40	50	60	70	80	90	100
Mantissa	.000	.301	.477	.602	.698	.778	.845	.903	.954	.000

600. The numbers 326500, 32650, 3265, 326.5, 32.65, 3.265, .3265, .03265, etc., expressed by the same sequence of figures, but differing in the position of the decimal point, may all be derived from any one of the set by multiplying or dividing by some power of 10. Hence *the logarithms of any two of these numbers can differ from each other only by the logarithm of some power of 10;* that is, by some integer, positive or negative, the mantissa remaining the same for all. Hence

Numbers expressed by the same sequence of significant figures have the same mantissa in their logarithms.

$$\text{Thus, } \log 326500 = \log (3265 \times 10^2) = \log 3265 + \log 10^2 = 3.51388 + 2 = 5.51388.$$

$$\log .03265 = \log (3265 \div 10^5) = \log 3265 - \log 10^5 = 3.51388 - 5 = \bar{2}.51388.$$

601. Since the increase or decrease of the characteristic by so many units manifestly corresponds to the shifting of the decimal point so many places to the right or left in the antilog, what has been shown above in regard to one series of antilogs and their logarithms will evidently hold good in regard to any other similarly related series.

602. It follows that *the characteristic of the logarithm of any number is equal to the number of places by which the left-hand digit of that number is distant from the units place; and that it is positive when the digit lies to the left, negative when it lies to the right, and zero when it is in the units place.*

Thus, (1) $\log 3625 = 3.51388$, (5) $\log .3625 = \bar{1}.51388$,
(2) $\log 362.5 = 2.51388$, (6) $\log .03625 = \bar{2}.51388$,
(3) $\log 36.25 = 1.51388$, (7) $\log .003625 = \bar{3}.51388$,
(4) $\log 3.625 = 0.51388$, and so on.

603. It may be adopted as a good rule, that a *positive* characteristic is numerically *less* by 1 than the number of figures before the decimal point of the given number; and that a

negative characteristic is numerically *greater* by 1 than the number of zeros after the decimal point and before the first significant figure. Thus, the characteristic of log 73582 is 4; the characteristic of log 4.965 is 0; and the characteristic of log .00053 is $\bar{4}$.

As the characteristic can thus always be supplied by inspection of the given number, it is accordingly omitted from the tables, which might more appropriately be called *tables of mantissas* than tables of logarithms.

Write down the characteristics of the logs of the following numbers, placing the minus sign over negative characteristics:

1. 7; 70; 70000; .7; .00007.
2. 23; 2.3; .23; 230; .023.
3. 5360; .536; 5.36; .00536.
4. 43892; 43.892; 43892000.

604. The mantissas in the tables herein given (pp. 319–321) may be regarded as those of the logarithms of all numbers from 1 to 1000. The mantissa of log 365, for example, is found in the line opposite 36, the first two figures of the antilog, and in the column headed by 5, the third figure. The mantissa thus found is .56229.

In the line opposite 36, in the columns headed by 0, 1, 2, ··· 9, we find the mantissas .55630, .55751, .55871, ··· .56703, belong respectively to the logs of 360, 361, 362, ··· 369. In the line opposite 40, the mantissa in the column headed 0 is not only the mantissa of log 400, but also of log 40, log 4, log .04, etc.

605. To find the logarithm of a number expressed by not more than three figures,

To the proper characteristic annex the mantissa found opposite the first two figures in the column headed by the third.

Thus,

(1) $\log 300 = 2.47712$; $\log 30 = 1.47712$; $\log 3 = 0.47712$;

(2) $\log .03 = \bar{2}.47712$; $\log 57 = 1.75587$; $\log 579 = 2.76268$;

(3) $\log 4.78 = 0.67943$; $\log 10.5 = 1.02119$; $\log 2390 = 3.37840$.

NOTE. Always write down the characteristic before seeking the mantissa.

Find the logarithms of the following numbers:

5. 2, 3, 4, .5, .6, .007, 80, 900, 90000.

6. 23, 340, 4500, 5.3, .67, .072, .0085.

7. 121, 306, 272000, 35.4, 4.62, .537, .00643.

8. .876, 7.65, 65.4, 543, 4320, 32100.

606. The method of finding the logarithms of numbers expressed by *more* than three figures is made evident by a single illustration. Since the number 362.45, which is greater than 362, is *less* than 363, the difference between log 362 and log 363 is *greater* than the difference between log 362 and 362.45. Let d denote the difference between log 362 and log 363; then log 362.45 will be log 362 increased by *less* than d.

Assuming that the logarithmic difference for a fraction of a unit will be the same fractional part of the difference for a whole unit, — an assumption generally true as far as two places of decimals, — we can find log 362.45 by multiplying d by .45, and adding the product to log 362. But log 362, as found in the tables, is 2.55871, and the difference between log 362 and log 363 is .00120, and hence log 362.45 is obtained as follows:

$$\log 362 = 2.55871$$
$$d \times .45 = .00120 \times .45 = \quad 54$$

whence $$\log 362.45 = 2.55925$$

Since the mantissa for log 36245 is the same as that for 362.45 (§ 600), we may find log 36245 by considering 45 a decimal, and proceeding in exactly the same way, except that we make the characteristic 4 instead of 2. Hence,

607. To find the logarithm of a number expressed by more than three figures,

To the logarithm of the number expressed by the first three figures, as found in the table, add the product of the corresponding

tabular difference multiplied by the number expressed by the remaining figures regarded as a decimal.

In the tables of logarithms given below, the difference between each mantissa and the next higher mantissa is printed over the first. These differences are called *tabular differences.*

9. Find log 4.7389.

$$\log 4.73 = 0.67486 \qquad d = 92$$
$$d \times .89 = .00092 \times .89 = \underline{\quad 82} \qquad \underline{.89}$$
whence $$\log 4.7389 = 0.67568 \qquad 828$$
$$\underline{736}$$
$$d' = 81.88$$

For convenience the tabular difference may be regarded as an integral number, and the product be formed as at the right above. We reject the 88, and since the first rejected figure is not less than 5, we consider the 81 as 82. Generally increase the fifth figure by 1 when the sixth figure of a mantissa is *more* than 5, and drop the sixth figure when it is *less* than 5.

10. Find log .00124.

$$\log .00124 = \bar{3}.09342$$
$$d \times .6 = 349 \times .6 \qquad \underline{\quad 209}$$
whence $$\log .00124 = \bar{3}.09551$$

Find the logarithms of the following numbers:

11. 96520; 8736; 764.5; 65.48; 5.321; .004512.

12. 35962; 2045.3; 156.78; 93.125; 8.6545.

13. .004567; .0056789; .06789; .78901.

14. .89012; 90.123; 123.45; 2345.6; 34567.

608. The antilog of a given logarithm is found by practically reversing the above process. For example, the antilog of 4.81491 is obtained (1) by finding the mantissa in the tables, and writing the corresponding antilog. The mantissa .81491 is found in line 65, column 3, and hence the antilog of said mantissa is 653. But (2) since the characteristic of the given logarithm is 4, there must be five (4 + 1) integral orders in the

antilog, and hence the antilog sought is 65300. If the given characteristic had been 2, the antilog would have been 653; and, if the characteristic had been 0, the antilog would have been 6.53; and so on.

609. To obtain the antilog of 2.89387, we find in the tables the *next lower* mantissa, which is .89376, in line 78, column 3. We write 783 for the first three figures of the antilog. The remaining figures are found as below:

Given mantissa, .89387
Next lower mantissa, .89376

Dividing by d or 55, $55)11.00(.20$

Annexing the quotient .20 to 783, and pointing so as to have three $(2+1)$ integral orders, we obtain 783.2 as the required antilog. If the characteristic had been $\bar{2}$, we should have placed one cipher $(-2+1)$ after the decimal point, thus obtaining .07832 as the antilog.

610. To find the antilog of a given logarithm,

I. *If the given mantissa is found in the tables, write down the figures of the corresponding antilog, and determine the position of the decimal point by the given characteristic.*

II. *If the exact given mantissa is not found in the tables, subtract from it the next lower mantissa, and divide the difference by the tabular difference of this lower mantissa. Annex the first two figures of the quotient to the antilog of the lower mantissa, and place the decimal point as determined by the given characteristic.*

Find the antilogs of the following logarithms:

15. 0.30103; 0.47712; 0.60206; 0.84510; 0.95424.
16. 1.47712; 2.69897; 3.77815; $\bar{3}$.90309; $\bar{2}$.95424.
17. 1.07918; 1.36173; 1.65321; 1.81954; 1.92428.
18. 2.13113; 2.51388; 2.77633; 1.86435; $\bar{1}$.96904.
19. 3.64098; 4.77541; 2.82816; 0.91172; $\bar{4}$.72652.

611. TABLE OF LOGARITHMS: WITH TABULAR DIFFERENCES.

N	0	1	2	3	4	5	6	7	8	9
	432	428	424	419	416	412	408	404	400	396
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743
	393	389	386	382	379	376	372	369	366	363
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555
	360	357	355	351	349	346	343	340	338	335
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059
	333	330	328	325	323	321	318	315	314	311
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301
	309	307	305	303	301	299	297	295	293	291
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319
	289	287	285	283	281	279	278	276	274	272
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140
	271	269	267	266	264	262	261	259	258	256
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789
	254	253	252	250	249	248	246	245	243	242
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285
	241	239	238	237	235	234	233	232	230	229
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646
	228	227	226	225	223	222	221	220	219	218
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885
	217	216	215	213	212	211	210	209	208	207
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015
	206	205	204	203	202	202	201	200	199	198
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044
	197	196	195	194	193	193	192	191	190	189
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984
	188	188	187	186	185	184	184	183	182	181
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840
	181	180	179	178	178	177	176	176	175	174
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620
	173	173	172	171	171	170	169	169	168	167
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
	167	166	166	165	165	164	163	162	162	161
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975
	161	160	159	159	158	158	157	156	156	156
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560
	155	154	154	153	152	152	151	151	151	150
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090
	149	149	149	148	147	147	147	146	145	145
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567
	145	144	143	143	143	142	142	141	141	140
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996
	140	139	139	139	138	138	137	137	136	136
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379
	136	135	134	134	133	133	133	132	132	131
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720
	131	131	130	130	129	129	129	129	128	128
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020
	127	127	126	126	126	126	125	125	125	124
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283
	124	123	123	123	123	122	122	121	121	121
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509
	121	120	120	119	119	119	119	118	118	117
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703
	117	117	117	116	116	116	115	115	115	114
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864
	114	114	114	113	113	113	112	112	112	112
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995
	112	111	110	110	110	110	109	109	109	109
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097

TABLE OF LOGARITHMS.

N	0	1	2	3	4	5	6	7	8	9
	108	108	108	107	107	107	107	107	106	106
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172
	106	106	105	105	105	105	105	104	104	104
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221
	103	103	103	103	102	102	102	102	102	101
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246
	101	101	101	100	100	100	99	99	99	99
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246
	99	98	98	98	98	97	97	97	97	96
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225
	96	96	96	96	95	95	95	95	95	95
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181
	94	94	94	94	94	94	93	93	93	93
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117
	92	92	92	92	91	91	91	91	91	90
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034
	90	90	90	90	89	89	89	89	89	89
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931
	88	88	88	88	87	87	87	87	87	87
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810
	87	86	86	86	86	86	86	85	85	85
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672
	85	85	85	85	84	84	84	84	84	84
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517
	83	83	83	83	83	83	82	82	82	82
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346
	82	82	82	81	81	81	81	81	81	81
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159
	80	80	80	80	80	80	79	79	79	79
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957
	79	79	79	78	78	78	78	78	78	78
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741
	77	77	77	77	77	77	77	77	77	76
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511
	76	76	76	76	76	75	75	75	75	75
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268
	75	75	75	74	74	74	74	74	74	74
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012
	74	74	73	73	73	73	73	73	73	72
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743
	72	72	72	72	72	72	72	71	71	71
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462
	71	71	71	71	71	71	70	70	70	70
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169
	70	70	70	70	70	70	69	69	69	69
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865
	69	69	69	69	69	68	68	68	68	68
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550
	68	68	68	67	67	67	67	67	67	67
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224
	67	67	67	66	66	66	66	66	66	66
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889
	66	66	66	65	65	65	65	65	65	65
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543
	65	65	65	64	64	64	64	64	64	64
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187
	64	64	64	64	63	63	63	63	63	63
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822
	63	63	63	63	63	62	62	62	62	62
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448

N	0	1	2	3	4	5	6	7	8	9
	62	62	62	62	62	62	61	61	61	61
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065
	61	61	61	61	61	61	61	61	60	60
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673
	60	60	60	60	60	60	60	60	60	60
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273
	59	59	59	59	59	59	59	59	59	59
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864
	59	59	58	58	58	58	58	58	58	58
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448
	58	58	58	58	58	57	57	57	57	57
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024
	57	57	57	57	57	57	57	57	57	56
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593
	56	56	56	56	56	56	56	56	56	56
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154
	56	56	56	55	55	55	55	55	55	55
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708
	55	55	55	55	55	55	55	54	54	54
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255
	54	54	54	54	54	54	54	54	54	54
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795
	54	54	53	53	53	53	53	53	53	53
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328
	53	53	53	53	53	53	53	52	52	52
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855
	52	52	52	52	52	52	52	52	52	52
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376
	52	52	52	51	51	51	51	51	51	51
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891
	51	51	51	51	51	51	51	51	51	51
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399
	50	50	50	50	50	50	50	50	50	50
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902
	50	50	50	50	50	50	50	49	49	49
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399
	49	49	49	49	49	49	49	49	49	49
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890
	49	49	49	49	49	48	48	48	48	48
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376
	48	48	48	48	48	48	48	48	48	48
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856
	48	48	48	48	47	47	47	47	47	47
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332
	47	47	47	47	47	47	47	47	47	47
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802
	47	47	47	47	46	46	46	46	46	46
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267
	46	46	46	46	46	46	46	46	46	46
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727
	46	46	46	46	46	45	45	45	45	45
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182
	45	45	45	45	45	45	45	45	45	45
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632
	45	45	45	45	45	45	44	44	44	44
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078
	44	44	44	44	44	44	44	44	44	44
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520
	44	44	44	44	44	44	44	44	44	43
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957

APPLICATIONS TO NUMERICAL PROCESSES.

1. Find the continued product of 875.43, 3.1416, and .00543.

$\log 875.43 = 2.94222$
$\log 3.1416 = 0.49715$
$\log .00543 = \bar{3}.73480$
$\text{Sum} = \overline{1.17417} = \log 14.933$, *Ans.* (§ 590)

2. Find the quotient of 72.9 by 545.37.

$\log \quad 72.9 = 1.86273$
$\log 545.37 = 2.73670$
$\text{Difference} = \bar{1}.12603 = \log .13367.$ (§ 591)

3. Find the 50th power of 1.05.

$\log 1.05 = 0.02119$
$\qquad\qquad 50$
$\text{Product} = \overline{1.05950} = \log 11.468\cdots.$ (§ 592)
$\therefore$ 11.468 is the 50th power of 1.05.

4. Find the 12th root of 365.

$\log 365 = 2.56229$
Dividing by 12 $\big\lfloor \overline{2.56229}$
$\text{Quotient} = 0.21352 = \log 1.6350\cdots.$ (§ 593)

5. Find the 5th root of .0365.

$\log .0365 = \bar{2}.56229.$
Add to $\log -3+3, \quad \bar{5}+3.56229.$
Dividing by 5 (§ 593), $\quad \bar{1}.71246.$
The antilog of $\bar{1}.71246 = .51574$, the required root.

612. When, as in Example 5, we have to divide a negative characteristic which is not an exact multiple of the divisor, we add to the negative characteristic as many negative units as will make it such a multiple, and prefix the same number of positive units to the mantissa. Thus, above, $\bar{2}.56229 = \bar{5}+3.56229$, and, dividing by 5, we obtain $\bar{1}.71246$, the antilog of which is .51574, the required root.

When negative numbers occur in computations, we proceed as if all were positive, deciding the sign of the result according to the rule of signs. Thus, to obtain the product of 53.6 and -3.975 by logarithms, we proceed as if both were positive, and give the negative sign to the result.

Find by logarithms the values of the following expressions:

6. $535 \times .342$.
7. 2.937×1.505.
8. $6.7354 \times .8925$.
9. $12345 \times .0053782$.
10. $.075 \times .3678 \div 73.251$.
11. $\dfrac{56.32}{9.365}$.
12. $\dfrac{8275.5}{43296}$.
13. $\dfrac{.5673 \times 25.05}{763 \times 365.47}$.
14. $(5.2)^6$.
15. $(1.0546)^{10}$.
16. $(.6054 \times .53798)^5$.
17. $\sqrt[3]{5.35}$.
18. $\sqrt[5]{1.0562}$.

613. Exponential equations (§ 569) are most readily solved by the aid of logarithms. When, for example, the a, l, r, of a geometrical progression, are given to find n, the value of n must be found from the equation $l = ar^{n-1}$; whence $r^{n-1} = l \div a$.

Taking the logarithms of both members of this equation,

$$(n-1)\log r = \log l - \log a\cdot$$

whence
$$n = 1 + \frac{\log l - \log a}{\log r}$$

By taking from the table the logarithms of the given quantities a, l, r, and performing the indicated operations, we obtain the value of n. If in the foregoing formula we suppose $a = 3$, $l = .00019683$, $r = .3$, we obtain

$$n = 1 + \frac{\log .00019683 - \log 3}{\log .3},$$

or
$$= 1 + \frac{\bar{4}.29409 - 0.47712}{\bar{1}.47712};$$

whence
$$n = 1 + \frac{\bar{5}.81697}{\bar{1}.47712} = 1 + 8 = 9.$$

19. Given $a = 162$, $l = \frac{2}{27}$, $r = \frac{1}{3}$: find n.

20. Given $a = \frac{3}{91}$, $s = 45$, $r = 4$: find n.

21. Given $a = 343$, $s = 400\frac{1}{7}$, $l = \frac{1}{7}$: find n.

22. Given $r = 5$, $s = 1562$, $l = 1250$: find n.

Solve the exponential equations

23. $11^x = 1331$.

24. $5^{2x} = 15625$.

25. $12^x = 41.57$.

26. $13^x = 14$.

27. $17^x = 91$.

28. $ab^x = c$.

29. $4^{x-3} = 6$.

30. $3^{x+2} = 2187$.

31. $3^{x^2-4x+4} = 1200$.

(*See also* § 689.)

BUSINESS FORMULAS.

614. Problems in percentage can in most cases be readily solved by the methods of arithmetic; but even in arithmetic there may be advantage in the use of general formulas to indicate the processes involved.* Such formulas have the advantage of being so related that they can be derived from each other, and thus be readily reproduced when needed. In algebra the solutions of all percentage problems may be indicated by formulas with the further advantage that they permit the easy use of logarithms, thus greatly facilitating computation.

Formulas for the more common processes in percentage, including simple interest, have already been given (§§ 285–289); and all that is further needed is the presentation of general formulas for compound interest and annuities, with examples illustrating the use of logarithms.

Compound Interest.

615. The formula for finding the amount of any given principal at *compound interest* for any time and any rate per cent may be obtained as follows:

Let p denote the principal, r the rate per cent, and n the time

* For methods of using percentage formulas in arithmetic, see White's New Complete Arithmetic.

in years. Since the amount of one dollar in one year is $1 + r$, the amount of p dollars in one year is $p(1+r)$; and the amount of $p(1 + r)$ dollars in *one* year is

$$p(1 + r) \times (1 + r) = p(1 + r)^2,$$

which is accordingly the amount of p dollars in *two* years. The amount of $p(1 + r)^2$ dollars in one year is

$$p(1 + r)^2 \times (1 + r) = p(1 + r)^3,$$

which is the amount of p dollars in *three* years; and so on. Hence the amount of p dollars in n years is $p(1 + r)^n$; that is, denoting the amount by a,

$$a = p(1 + r)^n. \qquad (B')$$

616. Any three of the four numbers denoted by a, p, r, n, being given, the fourth may be found from the above formula, which therefore suffices for the solution of all cases in compound interest.

If a, p, r, are given to find n, we have to solve the exponential equation

$$(1 + r)^n = \frac{a}{p};$$

whence $$n \log (1 + r) = \log a - \log p,$$

and $$n = (\log a - \log p) \div \log (1 + r).$$

1. In how many years will \$ 365 amount to \$ 500 at 4 % compound interest?

Here $a = 500$, $p = 365$, and $1 + r = 1.04$. Substituting the logarithm of these numbers for those of a, p, $1 + r$, above, we find

$$n = (2.69897 - 2.56229) \div .01703 = 8.025.$$

2. At what rate per cent will \$ 1500 at compound interest amount to \$ 1889.568 in 3 years?

3. What principal will amount to \$ 535.9572 in 3 years at 6 % compound interest?

4. Find a formula to show how long it will take p dollars to amount to mp dollars at r % compound interest.

617. If the interest is payable at shorter intervals than a year, say every half year, then, at r % per annum, the interest on one dollar for half a year is $\frac{r}{2}$; and at compound interest the amount of p dollars in n years, that is, in $2n$ half years, is $p\left(1+\frac{r}{2}\right)^{2n}$, the same as the amount of p dollars in $2n$ years at $\frac{r}{2}$ %. In the same way, if interest is payable m times a year, the amount of p dollars at r % is expressed by the formula

$$a=p\left(1+\frac{r}{m}\right)^{mn}.$$

Annuities.

618. To find the amount of an annuity accumulating any number of years, allowing compound interest,*

Let a denote the number of dollars in the annuity, n the number of years, $1+r$ the amount of a dollar for one year, and A the required amount. At the end of the first year a is due, and amounts at the end of the second year to $a(1+r)$; hence the whole sum due at the end of the second year is $a+a(1+r)=a[1+(1+r)]$. At the end of the third year the sum due is, in like manner,

$$a[1+(1+r)+(1+r)(1+r)]=a[1+(1+r)+(1+r)^2].$$

By proceeding in this way, we find that the amount due at the end of n years is

$$a[1+(1+r)+(1+r)^2+(1+r)^3+\cdots+(1+r)^{n-1}],$$

a G.P. whose first term is a, and ratio $1+r$; hence we obtain $A=a\frac{(1+r)^n-1}{1+r-1}=\frac{a}{r}[(1+r)^n-1]$.

* Annuities at simple interest have practically no existence.

619. To find the present value of an annuity to continue a certain number of years, allowing compound interest,

Let p denote the present value. The amount of p in n years, that is, $p(1+r)^n$, should be equal to the accumulated amount of the annuity in the same time; hence

$$p(1+r)^n = \frac{a}{r}[(1+r)^n - 1],$$

$$p = \frac{a}{r}\left(1 - \frac{1}{(1+r)^n}\right). \qquad (1)$$

If we suppose n to be indefinitely great, that is, the annuity to be perpetual, then $\frac{1}{(1+r)^n}$ becomes $\frac{1}{\infty}=0$, and the foregoing formula becomes $p = \frac{a}{r}$. (2)

620. Besides problems relating to annuities proper, these formulas may be applied to the solution of a great variety of problems concerning life insurance, value of estates, increase of capital, population, etc.

PROBLEMS.

1. How much should a man pay down to obtain a life annuity of $ 1000, his expectation of life being estimated at 20 years, and interest reckoned at 5%?

Here $a = 1000$, $n = 20$, $r = .05$. Substituting these values for a, n, r, in Formula (1), we obtain

$$p = \frac{1000}{.05}\left(1 - \frac{1}{(1.05)^{20}}\right).$$

By the aid of logarithms we find $\frac{1}{(1.05)^{20}} = .3769\cdots$;

hence $$p = \frac{1000}{.05}(1 - .3769) = 12462.$$ *Ans.* $12462.

2. What is the value of a farm that yields a net rental of $ 900, interest reckoned at 6%?

Here we have to find the present value of a perpetual annuity of $900. By Formula (2) above, we obtain

$$p = \frac{900}{.06} = 15000.$$ *Ans.* $15000.

3. What is the present value of an annuity of $600 for 25 years, interest reckoned at 5%?

4. What sum should be paid for a 15 years' lease of a property yielding a net annual profit of $500, interest at 6%?

5. By giving up smoking, a man saves $50 a year. How much does he thus save in 30 years, interest at 5%?

6. A man with a capital of $10,000 spends every year $150 more than his income. In how many years will his capital be consumed, interest reckoned at 8%?

7. A population of 1,000,000 has a steady annual increase of 3%. In how many years will it double itself?

8. A man with a capital of $20,000 adds to it yearly $200 besides the interest. What will be the amount of his capital in 20 years, interest reckoned at 10%?

9. Having borrowed $12,000 at 5%, how much should I pay every year so as to discharge the debt in 10 years?

10. In order to accumulate $3000, to cover the expenses of a son's college course, a father invested a certain sum at 5%, on each recurring anniversary of his son's birth until the 18th inclusive. How much did he invest yearly?

11. What annual payment will amount to $5000 in 21 years at 6% compound interest?

12. What is the value of an estate yielding $1500 net annual income, interest reckoned at $5\frac{1}{2}$%?

CHAPTER XXI.*

UNDETERMINED COEFFICIENTS AND APPLICATIONS.

621. It is sometimes desirable to obtain for a given algebraic expression an equivalent of a certain form; as, for example, a series in ascending powers of x. To this end, we first assume that the given expression is equal to a series of the required form, but having *undetermined coefficients.* We then proceed to determine the values of these coefficients by means of the principles exemplified below.

622. (1) If an equation of the form

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots \quad (1)$$

is such that for every value of x the equation is an identity (§ 624), then *the coefficients of the like powers of* x *in the two members are equal.*

For, since the equation is satisfied by any value of x, let $x = 0$. Then every term containing x equals 0, and the equation reduces to

$$A = A'.$$

Subtracting these equals from the members of Equation (1), we have

$$Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + D'x^3 + \cdots \quad (2)$$

Dividing each member of (2) by x, we have

$$B + Cx + Dx^2 + \cdots = B' + C'x + D'x^2 + \cdots. \quad (3)$$

* This and the following chapters are designed for more advanced classes, and especially for students who are preparing for higher institutions whose entrance requirements may include an elementary knowledge of one or more of the subjects treated.

Since Equation (3) must be satisfied by any value of x, let $x = 0$; and then the equation reduces to

$$B = B'.$$

In the same way we can prove that $C = C'$, $D = D'$, and so on.

623. (2) If an equation of the form

$$A + Bx + Cx^2 + Dx^3 + \cdots = 0 \qquad (4)$$

is true for every value of x, *the coefficients must each equal zero.*

For, since x may have any value, let $x = 0$, and then the equation reduces to

$$A = 0.$$

Then omitting A in (4), and dividing by x, we have

$$B + Cx + Dx^2 + \cdots = 0. \qquad (5)$$

Since x may have any value in (5), let $x = 0$, and then we obtain $B = 0$; and in the same way we can prove that $C = 0$, $D = 0$, and so on.

624. It can be shown that the foregoing principles (§§ 622, 623) apply only to equations that contain *convergent* series (§ 631, note), but the proof belongs to higher algebra.

Let us now consider several of the more important applications of these principles.

RESOLUTION OF FRACTIONS.

625. To resolve a fraction into *partial fractions* is to find the fractions whose algebraic sum is the given fraction. This is the converse of the process of adding fractions given in § 264.

For example, $\dfrac{3}{x-5} - \dfrac{2}{x+3} = \dfrac{x+19}{x^2-2x-15}$; and conversely, the fraction $\dfrac{x+19}{x^2-2x-15}$ being given, it is required to find the partial fractions of which this given fraction is the algebraic sum.

Having factored the denominator, we assume that

$$\frac{x+19}{x^2-2x-15}=\frac{A}{x+3}+\frac{B}{x-5}, \tag{1}$$

A and B being the numbers which we wish to determine.

Clearing (1) of fractions, we obtain

$$x+19=Ax-5A+Bx+3B,$$

or

$$x+19=(A+B)x-(5A-3B).$$

Equating the coefficients of the like powers of x (§ 622), we obtain

$$A+B=1, \tag{2}$$

$$5A-3B=-19; \tag{3}$$

two equations each containing A and B, from which we obtain

$$A=-2,\ B=3;$$

whence

$$\frac{x+19}{x^2-2x-15}=\frac{3}{x-5}-\frac{2}{x+3}.$$

626. Suppose, however, that it be required to resolve into partial fractions the expression $\frac{2x^2-13x-9}{x^2-2x-15}$.

If, having factored the denominator, we should assume that

$$\frac{2x^2-13x-9}{x^2-2x-15}=\frac{A}{x+3}+\frac{B}{x-5},$$

we would be led, on clearing of fractions, to the absurd result $2=0$. This difficulty may be avoided by first reducing the fraction $\frac{2x^2-13x-9}{x^2-2x-15}$ to a mixed number, and then resolving the fractional part, as shown below.

Hence, whenever the numerator of the fraction to be resolved is not of lower degree than the denominator, *reduce the given fraction to a mixed number, and then resolve the fractional part.*

Thus, taking the fraction given above,

$$\frac{2x^2-13x-9}{x^2-2x-15}=2+\frac{-9x+21}{x^2-2x-15}.$$

Assume $$\frac{-9x+21}{(x+3)(x-5)}=\frac{A}{x+3}+\frac{B}{x-5};$$

then, clearing of fractions,

$$-9x+21=Ax-5A+Bx+3B=(A+B)x-(5A-3B);$$

whence, equating, $A+B=-9$; $5A-3B=-21$;

whence $$A=-6,\ B=-3;$$

whence $$\frac{2x^2-13x-9}{x^2-2x-15}=2-\frac{6}{x+3}-\frac{3}{x-5}.$$

Resolve into partial fractions

1. $\dfrac{2x-13}{x^2-13x+40}$.
2. $\dfrac{2x+15}{x^2-15x+56}$.
3. $\dfrac{7x-23}{x^2-6x+5}$.
4. $\dfrac{4x-29}{x^2+3x-10}$.
5. $\dfrac{13x+5}{6x^2+5x+1}$.
6. $\dfrac{2x^2+2x-6}{x^3+5x^2+6x}$.
7. $\dfrac{2x^2+x+3}{x^2-1}$.
8. $\dfrac{3x^3+3x^2+2}{x^3-x}$.
9. $\dfrac{2x^2+21x+13}{4x^4-5x^2+1}$.
10. $\dfrac{3x^2+3x+18}{x^3-9x}$.
11. $\dfrac{8x+12}{x^2+6x+8}$.
12. $\dfrac{2x^2+x-1}{2x^2+x-3}$.

EXPANSION OF FRACTIONS INTO SERIES.

627. (1) Let $\dfrac{1+2x}{1-x-x^2}$ be the given fraction.

Assume $$\frac{1+2x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+\cdots. \qquad (1)$$

Multiplying each member of (1) by $1 - x - x^2$, we obtain

$$1 + 2x = A + B \left| \begin{matrix} x + C \\ -A \end{matrix} \right| \left| \begin{matrix} x^2 + D \\ -B \\ -A \end{matrix} \right| \left| \begin{matrix} x^3 + \text{ etc.} \\ -C \\ -B \end{matrix} \right|$$

Whence $A = 1$; $B - A = 2$, $\therefore B = 3$; $C - (A + B) = 0$,

$\therefore C = 4$; $D - (B + C) = 0$, $\therefore D = 7$; etc.

Substituting these values in the assumed series (1), we obtain $\frac{1 + 2x}{1 - x - x^2} = 1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + \cdots$, in which each coefficient after the second is equal to the sum of the preceding two coefficients, and hence the series can be continued indefinitely.

628. (2) Let $\frac{1 - 2x}{x^2 - 3x + 2}$ be the given fraction.

We see by inspection that the first term of the development must contain x^{-2}; and we accordingly assume that

$$\frac{1 - 2x}{x^2 - 3x + 2} = Ax^{-2} + Bx^{-1} + C + Dx + \cdots, \qquad (1)$$

and proceed as above.

Similarly, in other cases, we should first determine by inspection what power of x must occur in the first term of the development, and then assume the series accordingly.

The fraction $\frac{1 - 2x}{x^2 - 3x + 2}$ may be written in the form $\frac{1 - 2x}{2 - 3x + x^2}$, and then developed as in (1) in § 627.

Such developments as those here referred to may be obtained by three distinct methods: to wit, (1) by *simple division*, which may be employed as a means of proving the correctness of the results obtained by other methods; (2) by the method of *undetermined coefficients*, as exemplified above; and (3) by means of the *binomial formula*, as shown §§ 630, 631.

Expand into an infinite series

1. $\frac{1-2x}{1+3x}$.
2. $\frac{1}{1-2x+x^2}$.
3. $\frac{1-x}{1-2x-3x^2}$.
4. $\frac{1+x}{1-x+x^2}$.
5. $\frac{1}{3-x}$.
6. $\frac{1+x}{1-2x-x^2}$.
7. $\frac{1+x}{1+2x+3x^2}$.
8. $\frac{a+bx}{1+2x+3x^2}$.

BINOMIAL FORMULA.

629. It has been shown by successive multiplications of the binomial $a+b$ (§ 321), that, if n denotes any positive integer from 2 to 5 inclusive,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots. \qquad (1)$$

It remains to prove that this formula holds true for any positive integral value of n (§ 324).

Assume that Formula (1) is true for any positive integral value of n. Multiplying both members of (1) by $a+b$, we have

$$(a+b)^{n+1} = a^{n+1} + na^nb + \frac{n(n-1)}{1\cdot 2}a^{n-1}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \cdots$$

$$(1)\times(+b) \qquad a^nb + \qquad na^{n-1}b^2 + \qquad \frac{n(n-1)}{1\cdot 2}a^{n-2}b^3 + \cdots$$

Collecting terms, we obtain

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{1\cdot 2}a^{n-1}b^2 + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \cdots. \qquad (2)$$

Comparing (1) and (2), we see that the same law holds in both for the formation of the coefficients, and also for the exponents; and hence, if the formula holds true for the expansion of $(a+b)^n$, it holds true for the expansion of $(a+b)^{n+1}$. But it has been shown by actual multiplication that Formula (1) holds true when $n=5$ (§ 321), and hence it holds true when $n=6$; and, if it holds true when $n=6$, it holds true when $n=7$, and so on indefinitely *to any power denoted by a positive integral exponent.*

Again, if b is a negative number, then, since the odd powers of a negative number are negative, and the even powers positive (§ 318), we evidently have, from (1),

$$(a+b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots, \qquad (3)$$

the alternate signs being + and −.

Equations (1) and (3) can be written as one formula as follows:

$$(a \pm b)^n = a^n \pm na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots. \qquad (A)$$

Negative or Fractional Exponents.

630. It can be shown that the above formula (A) holds also for both negative and fractional exponents; but the results obtained are reliable only when the resulting infinite series is *convergent.*

When the exponent n is a positive integer, the series terminates with $n+1$ terms; for the coefficient of the next and each succeeding term contains the zero factor $n-n$, which causes the terms to vanish. But when n is negative or fractional, no factor can become zero, and hence the series will

never terminate, but is an *infinite series, either convergent or divergent.*

631. The convergency and divergency of series is too difficult a subject for consideration in an elementary algebra, and the same is true of a perfectly rigorous demonstration of the binomial formula for negative and fractional exponents.*

* None of the proofs given in the ordinary algebras are free from objections; and only a few of the more recent make any reference to failure of the formula in the case of *divergent* infinite series.

We here add a few notes respecting the convergency and divergency of series to indicate the difficulty of the subject.

A series consisting of an infinite number of terms, which succeed each other according to some fixed law, is said to be *convergent* when the sum of its first n terms approaches nearer and nearer to a finite limiting value, according as n is taken greater and greater; and this limiting value is called *the sum of the series*, and from this value the series can be made to differ by an amount less than any assigned quantity on taking a sufficient number of terms.

If the sum of the first n terms approximates to no finite limit, the series is said to be *divergent.*

Let us illustrate by a few examples.

Take $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

If $x < 1$, this series approaches a fixed limit. If, for example, $x = \frac{1}{3}$, then we have

$$\frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots.$$

The greater the number of terms taken, the nearer the sum approaches the fixed limit $\frac{3}{2}$, and similarly for any value of x less than 1. Hence the series is *convergent* if $x < 1$.

On the other hand, if $x > 1$, the series is *divergent.* For example, let $x = 2$, and then we have the manifestly absurd result

$$\frac{1}{1-2} = -1 = 1 + 2 + 4 + 8 + \cdots.$$

In this case the sum does not approach a fixed limit, and the series is therefore divergent:

Again, let $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

We shall therefore assume the formula to be true for such exponents in cases in which the values of a and b are such as to result in *convergent* series; and, in the examples given below, care will be taken to indicate when the resulting series is convergent.

1. Expand $(a+b)^{-1}$.

Here $n=-1$; $\frac{n-1}{2}=\frac{-1-1}{2}=-1$; $\frac{n-2}{3}=-1$; $\frac{n-r}{r+1}=-\frac{1+r}{r+1}=-1$.

It is thus seen that the coefficients are alternately $+1$ and -1, since the product of an even number of negative factors is $+$, and of an odd number of factors $-$ (§ 106).

Hence

$$(a+b)^{-1}=a^{-1}-a^{-2}b+a^{-3}b^2\cdots+a^{-r}b^{r-1}=\frac{1}{a}\left(1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\cdots\right),$$

a series which is convergent when $a>b$.

It is often found convenient, as in this example, to place a factor outside a parenthesis, and to change negative to positive exponents.

2. Expand $(a+b)^{\frac{1}{2}}$.

Using formula A, b being positive, we have

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}b-\frac{1}{2\cdot4}a^{-\frac{3}{2}}b^2+\frac{1\cdot3}{2\cdot4\cdot6}a^{-\frac{5}{2}}b^3-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}a^{-\frac{7}{2}}b^4+\cdots$$

$$=\sqrt{a}\left(1+\frac{1}{2}\frac{b}{a}-\frac{1}{2\cdot4}\frac{b^2}{a^2}+\frac{1\cdot3}{2\cdot4\cdot6}\frac{b^3}{a^3}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\frac{b^4}{a^4}+\cdots\right).$$

This series is also convergent when $a>b$.

It is usually well, as in this series, to keep the factors of the coefficients separate, so as to show what is termed the *law of the series*.

Here, if $x<1$, the sum of the series approaches a fixed limit, and hence is convergent.

If $x=1$, then the series becomes $1-1+1-1+\cdots$. For an even number of terms the sum is 0; but for an odd number of terms the sum is $+1$. In this case the sum oscillates from 0 to $+1$ without approaching a fixed limit, and hence the series is divergent.

If $x>1$, the sum of an even number of terms of the series is always *negative*, and the sum of an odd number of terms is always *positive*. Hence, in this case, the sum does not approach a fixed limit, and the series is therefore divergent.

3. Expand $\frac{1}{\sqrt{a+b}}$ or $(a+b)^{-\frac{1}{2}}$.

Here $n=-\frac{1}{2}$; $\frac{n-1}{2}=\frac{-\frac{1}{2}-1}{2}=-\frac{3}{4}$; $\frac{n-2}{3}=-\frac{5}{6}$; $\frac{n-3}{4}=-\frac{7}{8}$; and so on

Hence $(a+b)^{-\frac{1}{2}}=a^{-\frac{1}{2}}-\frac{1}{2}a^{-\frac{3}{2}}b+\frac{1\cdot3}{2\cdot4}a^{-\frac{5}{2}}b^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}a^{-\frac{7}{2}}b^3+\cdots$

$$=\frac{1}{\sqrt{a}}\left(1-\frac{1}{2}\frac{b}{a}+\frac{1\cdot3}{2\cdot4}\frac{b^2}{a^2}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{b^3}{a^3}+\cdots\right).$$

This series is convergent when $a>b$.

Expand to four terms

4. $(1+\frac{1}{16})^{-\frac{1}{4}}$.
5. $(1-\frac{1}{5})^{-\frac{1}{3}}$.
6. $(1+\frac{1}{5})^{\frac{1}{4}}$.
7. $(2+\frac{1}{4})^{\frac{1}{3}}$.
8. $(1+a)^{\frac{1}{2}}$.
9. $(1-a)^{\frac{2}{3}}$.
10. $(a+b)^{-3}$.
11. $(a-b)^{-\frac{2}{3}}$.
12. $(1+\frac{1}{5})^{\frac{1}{5}}$.

The series in 8 and 9 is convergent when $a<1$, and in 10 and 11 when $a>b$.

Extraction of Roots by the Binomial Formula.

632. The binomial formula may be employed for the extraction of any roots of a given number N.

For $\sqrt[n]{N}=\sqrt[n]{a^n\pm b}=a\left(1\pm\frac{b}{a^n}\right)^{\frac{1}{n}}$, a^n being the nth power which is near to N, either greater or less than N; but $\frac{b}{a^n}$ must be a proper fraction, otherwise the expansion will be a *divergent series*, and the result will not give the correct root.

When $\left(1-\frac{b}{a^n}\right)^{\frac{1}{n}}$ is expanded, the number of terms required to give a certain degree of approximation will depend upon the relative value of a^n and b.

As an example, let us find the square root of 72.

$$\sqrt{72}=\sqrt{64+8}=8\,(1+\tfrac{1}{8})^{\frac{1}{2}}, \text{ since } \frac{b}{a^n}=\frac{8}{64}=\frac{1}{8}.$$

Expanding $(1+\frac{1}{8})^{\frac{1}{2}}$ by Formula (A), we have (since $a=1$, $b=\frac{1}{8}$, and $n=\frac{1}{2}$),

$$8\left(1+\frac{1}{8}\right)^{\frac{1}{2}}=8\left(1+\frac{1}{2}\cdot\frac{1}{8}-\frac{1}{2\cdot4\cdot8^2}+\frac{1\cdot3}{2\cdot4\cdot6\cdot8^3}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8\cdot8^4}+\cdots\right).$$

Performing the operations indicated, we obtain

$$\sqrt{72}=8.48528,$$

which is correct to the fifth decimal figure.

But few terms are necessary in this example, since a^n is 8 times b, and the convergence of the series is rapid.

1. Extract the cube root of 17.

Here the third power nearest to 17 is 8, whence

$$\sqrt[3]{17}=\sqrt[3]{8+9}=2\,(1+\tfrac{9}{8})^{\frac{1}{3}}.$$

But since $\frac{b}{a^n}$ is an improper fraction, the expansion of $(1+\frac{9}{8})^{\frac{1}{3}}$ will give a *divergent series*, and the result will not be a correct value of $\sqrt[3]{17}$.

But since $\sqrt[3]{17}=\sqrt[3]{27-10}=3\sqrt[3]{1-\frac{10}{27}}$, we can write

$$\sqrt[3]{17}=3\left(1-\frac{10}{27}\right)^{\frac{1}{3}}=3\left(1-\frac{1}{3}\cdot\frac{10}{27}-\frac{2}{3\cdot6}\left(\frac{10}{27}\right)^2-\frac{2\cdot5}{3\cdot6\cdot9}\left(\frac{10}{27}\right)^3-\frac{2\cdot5\cdot8}{3\cdot6\cdot9\cdot12}\left(\frac{10}{27}\right)^4-\cdots\right).$$

This is a convergent series, and, by taking a sufficient number of terms, we can find the value of $\sqrt[3]{17}$ to any required approximation. The convergence of the series is, however, not rapid.

633. The following examples will converge rapidly, and but a few terms will be required to obtain results correct to five or six places of decimals.

2. $\sqrt[3]{28}$. 3. $\sqrt[5]{34}$. 4. $\sqrt[7]{129}$. 5. $\sqrt[5]{244}$.

This process of extracting the roots of numbers, though interesting as an application of the binomial formula, is practically of little value when logarithmic tables are available.

CHAPTER XXII.

DETERMINANTS.

634. An expression whose value depends on two or more quantities is said to be a **function** of those quantities. Thus, $ab + bc$ is a function of a, b, and c (§ 540).

In solving a system of equations of the first degree, also defined as *linear*, a class of functions appear, called **determinants**, first observed by Leibnitz in 1693.

635. Let us take the equations

$$\begin{cases} a_1x + b_1y = 0, & (1) \\ a_2x + b_2y = 0. & (2) \end{cases}$$

Multiplying (1) by b_2 and (2) by $-b_1$, adding, and then dividing the resulting equation by x, we have

$$a_1b_2 - a_2b_1 = 0.$$

The expression $a_1b_2 - a_2b_1$ is a *determinant*. It is usually written with vertical lines at the left and right, called the *square form*, thus: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. Hence $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$.

636. The first member of this identity is the *undeveloped* form of the determinant, and the second member its *developed* form.

The numbers a_1, a_2, b_1, and b_2 are called the **elements** of the determinant, and the expressions a_1b_2 and a_2b_1 in the developed form are called its **terms**.

The horizontal lines of letters in the undeveloped form of a determinant are called **rows**, and the vertical lines of letters are called **columns**.

The determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has two rows and two columns, and, in the developed form $(a_1b_2 - a_2b_1)$ each term is the product of two factors; hence the determinant is said to be *of the second order.*

It will be seen later that in every determinant there are as many rows as columns, and as many of each as there are elements in each term.

637. It will be observed that the developed form of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the product of the elements in the diagonal from the upper left-hand corner to the lower right-hand corner (a_1b_2) *minus* the product of the elements in the diagonal from the lower left-hand corner to the upper right-hand corner $(a_2\, b_1)$.

The first of these diagonals $(a_1\, b_2)$ is called the *principal* diagonal, and the second (a_2b_1) the *secondary* diagonal.

638. Since $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1 \equiv a_1b_2 - b_1a_2$, it follows that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Hence the value of the determinant is not altered *by changing the rows into columns, and the columns into rows.*

639. Again, since $a_1b_2 - a_2b_1 \equiv -(b_1a_2 - b_2a_1) \equiv -(a_2b_1 - a_1b_2)$, it follows that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} \equiv -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}.$$

Hence, if the order of the columns or rows be changed, *the sign of the determinant will be changed, but its absolute value will not be altered.*

It will be seen later (§ 654) that the above laws hold true for a determinant *of any order.*

640. Let us now solve the simultaneous linear equations

$$\begin{cases} a_1x + b_1y = c_1, & (1) \\ a_2x + b_2y = c_2. & (2) \end{cases}$$

Multiplying (1) by b_2 and (2) by $-b_1$, and adding the resulting equations, we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}.$$

Multiplying (1) by $-a_2$, and (2) by a_1, and adding the resulting equations, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

It will be observed that the numerators and the denominators in the values of x and y are *determinants*, the denominators being alike, and hence we can write

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad (3) \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad (4)$$

Comparing the numerators and denominators, it will be seen that the numerator of (3) can be obtained from the denominator by changing the column of a's in the denominator to c's; and, in like manner, that the numerator of (4) can be obtained from the denominator by changing the column of b's in the denominator to c's.

Exercises.

1. Solve, by means of (3) and (4), the equations

$$\begin{cases} 5x + 2y = 19, \\ 3x + 4y = 17. \end{cases}$$

Here $a_1 = 5$, $a_2 = 3$; $b_1 = 2$, $b_2 = 4$; $c_1 = 19$, and $c_2 = 17$. Substituting these values in (3) and (4), we obtain

$$x = \frac{\begin{vmatrix} 19 & 2 \\ 17 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{76 - 34}{20 - 6} = 3; \quad y = \frac{\begin{vmatrix} 5 & 19 \\ 3 & 17 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{85 - 57}{20 - 6} = 2.$$

Solve in like manner the following linear equations:

2. $\begin{cases} 5x+2y=12, \\ 3x+4y=10. \end{cases}$ 4. $\begin{cases} 9x+y=10, \\ 5x-2y=3. \end{cases}$

3. $\begin{cases} 4x-y=18, \\ 2x+3y=16. \end{cases}$ 5. $\begin{cases} ax+by=c, \\ dx+cy=f. \end{cases}$

Expand the following determinants:

6. $\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix}$. 7. $\begin{vmatrix} 15 & 0 \\ 3 & 5 \end{vmatrix}$. 8. $\begin{vmatrix} 4 & 5 \\ 3 & 0 \end{vmatrix}$.

DETERMINANTS OF THE THIRD ORDER.

641. Let us take the equations

$$\begin{cases} a_1x+b_1y+c_1z=0, & (1) \\ a_2x+b_2y+c_2z=0, & (2) \\ a_3x+b_3y+c_3z=0. & (3) \end{cases}$$

Multiplying (1) by $b_2c_3-b_3c_2$, (2) by $-b_1c_3+b_3c_1$, and (3) by $b_1c_2-b_2c_1$, and adding the resulting equations, thus eliminating y and z, and then dividing the resulting equation by x, we obtain

$$a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)=0, \qquad (4)$$

or
$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}=0. \qquad (5)$$

If the operations indicated in (4) be performed, we have

$$a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2c_1b_3+a_3b_1c_2-a_3c_1b_2\equiv$$
$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \qquad (6)$$

642. The first member of (6) is a determinant; and, since each term contains *three elements*, it is called a determinant *of the third order.*

Such a determinant is usually written in the square form, and hence (6) may be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\equiv a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \qquad (7)$$

643. A determinant of the third order may be readily developed as follows:

Repeat the first and second rows below, as at the right.

For the *positive* terms, begin with a_1, a_2, a_3, respectively, and multiply the elements diagonally *downward*, obtaining $a_1b_2c_3$, $a_2b_3c_1$, $a_3b_1c_2$.

For the *negative* terms, begin with a_3, a_1, a_2, respectively, and multiply the elements diagonally *upward*, obtaining $a_3b_2c_1$, $a_1b_3c_2$, $a_2b_1c_3$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}$$

The development is $a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$.

In practice it will not be necessary to write down the repeated rows, for the work can easily be done mentally.

644. If we change the rows to columns, and the columns to rows, we shall obtain the same result. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3. \quad (8)$$

It thus appears that the value of a determinant of the third order is not altered *by changing the rows to columns, and the columns to rows.*

645. If the order of any two rows or columns be interchanged, the result obtained will be the negative of (8). Thus,

$$\begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} \equiv a_2b_1c_3 + b_2c_1a_3 + c_2a_1b_3 - c_2b_1a_3 - b_2a_1c_3 - a_2b_3c_1.$$

It is thus seen, that, if any two columns or rows of a determinant of the third order be interchanged, *the sign of the determinant will be changed, but its value will not be altered.*

646. It appears from the first member of (6), § 641, that a determinant of the third order consists of *six* terms, three of which are *positive,* and three *negative.*

It is seen from (7) that a determinant of the third order can be resolved into *the sum of three determinants of the second order.*

EXERCISES.

Develop the following determinants:

1. $\begin{vmatrix} 3 & 2 & 5 \\ 1 & 3 & 2 \\ 4 & 1 & 3 \end{vmatrix} = -24.$

2. $\begin{vmatrix} 5 & 1 & 4 \\ 3 & 2 & 5 \\ 4 & -1 & 2 \end{vmatrix} = 15.$

3. $\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = -360.$

4. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 5 & -2 \\ -4 & 3 & 2 \end{vmatrix} = 86.$

5. $\begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 2 & 1 \end{vmatrix} = 3b - 5a - 2c.$

6. $\begin{vmatrix} a & b & c \\ 1 & 3 & 2 \\ 2 & 4 & 1 \end{vmatrix} =$ what?

7. Show that $\begin{vmatrix} 8 & 4 & 5 \\ 2 & 3 & 1 \\ 5 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 5 \\ 4 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix}.$

647. Let us now solve the linear equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3. & (3) \end{cases}$$

Multiplying (1) by $b_2c_3 - b_3c_2$, (2) by $-b_1c_3 + b_3c_1$, and (3) by $b_1c_2 - b_2c_1$, and adding the resulting equations, the terms that contain y and z vanish, and we obtain

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 - d_2b_1c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}. \quad (4)$$

Similarly, multiplying (1) by $-a_2c_3 + a_3c_2$, (2) by $a_1c_3 - a_3c_1$, and (3) by $-a_1c_2 + a_2c_1$, and adding the resulting equations, the terms that contain x and z vanish, and we have, after arranging the terms,

$$y = \frac{a_1d_2c_3 - a_1d_3c_2 - a_2d_1c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}. \quad (5)$$

Lastly, multiplying (1) by $a_2b_3 - a_3b_2$, (2) by $-a_1b_3 + a_3b_1$, and (3) by $a_1b_2 - a_2b_1$, and adding the resulting equations, the terms that contain x and y vanish, and we have, after rearranging the terms,

$$z = \frac{a_1b_2d_3 - a_1b_3d_2 - a_2b_1d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}. \qquad (6)$$

The numerator and the denominator in the above values of x, y, and z respectively are determinants of the third order, and they may be written in square form, thus:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} (4') \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} (5') \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} (6')$$

It is seen that the denominators in these values of x, y, and z are the *same;* and that the numerator of the value of x can be obtained from its denominator by changing the column of a's in the denominator to d's; that the numerator of the value of y can be obtained from its denominator by changing the column of b's in the denominator to d's; and that the numerator of the value of z can be obtained from its denominator by changing the column of c's in the denominator to d's.

8. Solve the linear equations $\begin{cases} 3x + 4y + 2z = 17, \\ 5x + \ y + 3z = 16, \\ 4x + 3y + 7z = 31. \end{cases}$

Here $a_1 = 3$, $a_2 = 5$, $a_3 = 4$; $b_1 = 4$, $b_2 = 1$, $b_3 = 3$; $c_1 = 2$, $c_2 = 3$, $c_3 = 7$; $d_1 = 17$, $d_2 = 16$, $d_3 = 31$. Substituting these values in (4'), (5'), and (6'), we obtain

$$x = \frac{\begin{vmatrix} 17 & 4 & 2 \\ 16 & 1 & 3 \\ 31 & 3 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & 2 \\ 5 & 1 & 3 \\ 4 & 3 & 7 \end{vmatrix}} = \frac{119 + 96 + 372 - 62 - 448 - 153}{21 + 30 + 48 - 8 - 140 - 27} = \frac{-76}{-76} = 1.$$

$$y = \frac{\begin{vmatrix} 3 & 17 & 2 \\ 5 & 16 & 3 \\ 4 & 31 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & 2 \\ 5 & 1 & 3 \\ 4 & 3 & 7 \end{vmatrix}} = \frac{-152}{-76} = 2. \qquad z = \frac{\begin{vmatrix} 3 & 4 & 17 \\ 5 & 1 & 16 \\ 4 & 3 & 31 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & 2 \\ 5 & 1 & 3 \\ 4 & 3 & 7 \end{vmatrix}} = \frac{-228}{-76} = 3.$$

The values of x, y, and z may be written as determinants directly, without substituting the values of the elements. The denominator in each determinant should be written first, and then the numerator, according to the direction given above.

Solve the following linear equations:

9. $\begin{cases} 3x - 2y - z = 4, \\ 5x - 3y + z = 10, \\ 2x + 4y - 3z = 11. \end{cases}$

11. $\begin{cases} 5x - 3y + z = 16, \\ 9x + 2y - 3z = 14, \\ x - 4y - 5z = 10. \end{cases}$

10. $\begin{cases} 2x - 4y + 3z = 10, \\ 3x + y - 2z = 6, \\ x + 3y - z = 20. \end{cases}$

12. $\begin{cases} x + y + z = 6, \\ 5x + 4y + 3z = 22, \\ 3x + 4y - 3z = 2. \end{cases}$

DETERMINANTS OF ANY ORDER.

648. A determinant with n rows and n columns is called a determinant *of the* n*th order*. It contains n^2 elements.

A determinant of the nth order may be developed by taking the sum, with the proper sign (to be established later), of all the possible products of its n^2 elements that can be formed by taking as factors one element, and only one, from each row, and one, and only one, from each column. Each term of the developed determinant will contain n elements

Take, for example, the determinant: $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

This is a determinant of the fourth order; and the sum of all possible products of its 16 elements in sets of 4, one being taken from each row and one from each column, will be its development or value.

649. The product of the elements in the principal diagonal from a_1 to d_4, which is $a_1b_2c_3d_4$, is taken as the *leading term.* Its subscripts are in natural order from left to right (1, 2, 3, 4), and it is regarded as *positive.*

The signs of the other terms are determined by the number of *interchanges* required to bring the subscripts in each in their natural order. If the number of interchanges is *even,* the term is *positive;* if the number of interchanges is *odd,* the term is *negative.*

Thus, in $a_2b_1c_4d_3$, to bring 1 to the first place, it must be interchanged with 2, and to bring 3 to the third place, it must be interchanged with 4, making *two* interchanges, and hence the term is *positive.* But in $a_1b_4c_3d_2$, the 2 must be interchanged with 3 and with 4, and 3 must be interchanged with 4, making *three* interchanges, and hence the term is *negative.*

650. A determinant may be indicated by its leading term in vertical lines. Thus, $|a_1\ b_2|$ denotes a determinant of the second order; $|a_1\ b_2\ c_3|$, a determinant of the third order; $|a_1\ b_2\ c_3\ d_4|$, a determinant of the fourth order; and $|a_1\ b_2\ c_3\ d_4 \cdots r_n|$, a determinant of the nth order. A determinant may also be expressed by the notation $\Sigma \pm$ before its leading term, as $\Sigma \pm a_1b_2c_3$.

MINORS AND COFACTORS.

651. If we strike out of a determinant any number of rows and the same number of columns, the remaining elements form a determinant called a **minor.** This minor, and the determinant formed by the elements common to the rows and columns stricken out, are called **cofactors.**

Take, for example, the determinant (1), and strike out the first row and the first column as in (2). Then (3) is its first minor, and a_1 is its cofactor.

(1) $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ (2) $\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} & \cancel{d_1} \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ (3) $\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$ (4) $\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} & \cancel{d_1} \\ \cancel{a_2} & \cancel{b_2} & \cancel{c_2} & \cancel{d_2} \\ \cancel{a_3} & \cancel{b_3} & \cancel{c_3} & \cancel{d_3} \\ \cancel{a_4} & \cancel{b_4} & \cancel{c_4} & \cancel{d_4} \end{vmatrix}$

If we strike out the first and second rows and the first and second columns, as in (4), then $\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$ is the minor, and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ its cofactor, or corresponding minor.

652. From the determinant (1) above, we can obtain the cofactors.

$$\begin{array}{cccc} (1) & (2) & (3) & (4) \\ a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} & a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} & a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} & a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \end{array}$$

It will be seen that (1) contains all the possible terms that contain a_1; (2), all the possible terms that contain a_2; (3), all the possible terms that contain a_3; and (4), all the possible terms that contain a_4. But in each one of these minor determinants there are *six* terms, as shown in § 646, and hence there are 24 terms (6×4) in a determinant of the fourth order.

In like manner it can be shown that there are 120 terms, (24×5) or ($1 \times 2 \times 3 \times 4 \times 5$), in a determinant of the fifth order; and it may be shown that the number of terms in a determinant of the nth order is the continued product of $1 \times 2 \times 3 \times 4 \cdots \times n$.

653. It further appears from the above that the expressions

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \quad a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \quad a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix}, \text{ and } a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix},$$

taken together, comprise all the terms in the original determinant (1) in § 651.

It is evident that the first is *positive*, since, when each term of the minor is multiplied by a_1, the order of the subscripts is not changed. The second is *negative*, since, when each term of the minor is multiplied by a_2, *one* interchange is necessary to bring a_2 to the second place. The third is *positive*, since, when each term is multiplied by a_3, *two* interchanges are necessary to bring a_3 to the third place. The fourth is *negative*, since, when each term is multiplied by a_4, *three* interchanges are necessary to bring a_4 to the fourth place.

Hence the following identity:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

It is evident from the foregoing considerations that a determinant of the nth order can be made to depend on n determinants of the order of $n-1$.

PROPERTIES.

654. Let us here recapitulate the properties of determinants already established, and add a few others which are of special value in manipulating determinants.

1. *The value of a determinant is not altered by changing the rows to columns, and the columns to rows.*

This has been shown to be true of determinants of the second and third orders (§§ 638, 643). It is also true of determinants of any order, since the succession of terms is the same in the changed form as in the original determinant.

2. *If any two rows or columns be interchanged, the sign of the determinant will be changed, but its value will not be altered.*

This has also been shown to be true of determinants of the second and third orders (§§ 639, 645). It is generally true; for, if the rows interchanged be *adjacent*, the changed form will require one more interchange than the original form to bring the subscripts into regular order; and, if the rows interchanged are *not* adjacent, an odd number of interchanges will be necessary.

3. *If two rows or columns of a determinant are identical, the determinant vanishes.*

For let D be the value of the determinant. Then, if the identical rows be interchanged, the sign of the determinant will be changed by Property 2, but its value will not be altered. Hence $D=-D$. $\therefore\ 2D=0$. $\therefore\ D=0$.

4. *If each element in any row or column be multiplied by the same number, the determinant will be multiplied by that number.*

This is evident from the fact that every term of a determinant contains *one*, and *only one*, element from the same row or column.

Hence $$\begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = ma_1b_2 - ma_2b_1 = m(a_1b_2 - a_2b_1).$$

5. *If the elements of any row are like multiples of any other row, the determinant vanishes.*

For $$\begin{vmatrix} ma_1 & a_1 & b_1 \\ ma_2 & a_2 & b_2 \\ ma_3 & a_3 & b_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0.$$

The determinant is put in the second form by Property 4, and this in turn vanishes by Property 3.

6. *A determinant of the* n*th order can be expressed in* n *determinants of the* $(n-1)$*th order.*

This has been proved in § 653.

7. *If all the elements but one of any row are zeros, the order of the determinant is reduced by one.*

For if A_1, A_2, A_3, A_4, etc., be the minors corresponding to a_1, a_2, a_3, a_4, etc., and if D denotes the determinant, then

$$D = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4 + \text{etc.}$$

Now, let D_1 represent the determinant when all the elements a_1, a_2, a_3, a_4, etc., except one (say a_1), are equal to zero, then $D_1 = a_1A_1$. But the order of D_1, which is the same as that of A_1, is one less than D: hence the property is proved.

8. *If every element of any column or row is a* SUM *of two or more numbers, the determinant is equal to the* SUM *of two or more determinants of the same order.*

For $$\begin{vmatrix} a_1 + k + l & b_1 & c_1 \\ a_2 + m + n & b_2 & c_2 \\ a_3 + s + p & b_3 & c_3 \end{vmatrix}$$

$$= (a_1 + k + l) \begin{vmatrix} b_2 + c_2 \\ b_3 \end{vmatrix} - (a_2 + m + n) \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3 + s + p) \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + k \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - m \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

$$+ s \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + l \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - n \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + p \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k & b_1 & c_1 \\ m & b_2 & c_2 \\ s & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} l & b_1 & c_1 \\ n & b_2 & c_2 \\ p & b_3 & c_3 \end{vmatrix}.$$

9. *The value of a determinant is not altered by adding to or subtracting from the elements of any column the corresponding elements of the other columns multiplied by the same factor.*

For $\begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} + m \begin{vmatrix} b_1b_1c_1 \\ b_2b_2c_2 \\ b_3b_3c_3 \end{vmatrix} + n \begin{vmatrix} c_1b_1c_1 \\ c_2b_2c_2 \\ c_3b_3c_3 \end{vmatrix}$.

But by Property 5 the last two determinants vanish, and hence

$$\begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In like manner it may be shown that the value of a determinant is not altered by subtracting elements as stated in Property 9.

This property holds true for a determinant of any order.

10. *If the signs of all the elements of any column or row be changed, the sign of the determinant will be changed.*

For this changes the signs of each term of the determinant.

655. The application of one or more of the above properties will make apparent the changes in the following determinants:

$$\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = \begin{vmatrix} 15 & 1 & 6 \\ 15 & 5 & 7 \\ 15 & 9 & 2 \end{vmatrix} = 15 \begin{vmatrix} 1 & 1 & 6 \\ 1 & 5 & 7 \\ 1 & 9 & 2 \end{vmatrix} = 15 \begin{vmatrix} 1 & 1 & 6 \\ 0 & 4 & 1 \\ 0 & 8 & -4 \end{vmatrix} = 15 \begin{vmatrix} 4 & 1 \\ 8 & -4 \end{vmatrix}$$

$$= 15(-16-8) = -15 \times 24 = -360.$$

The second determinant is obtained from the first by adding the second and third columns to the first, agreeable to Property 9. The third determinant is obtained from the second according to 4. The fourth is obtained from the third by subtracting the first row from the second and third rows respectively, according to 9. The fifth is obtained from the fourth according to 7. Then the determinant of the second order is developed.

Trace the following changes in the same determinant:

$$\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = \begin{vmatrix} 15 & 15 & 15 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = 15 \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = 15 \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \\ 4 & 5 & -2 \end{vmatrix} = 15 \begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix}$$

$$= 15(-4-20) = -15 \times 24 = -360.$$

Trace also the following changes:

$$\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} = \begin{vmatrix} 8 & 1 & 6 \\ 15 & 15 & 15 \\ 4 & 9 & 2 \end{vmatrix} = 15 \begin{vmatrix} 8 & 1 & 6 \\ 1 & 1 & 1 \\ 4 & 9 & 2 \end{vmatrix} = 15 \begin{vmatrix} 7 & 1 & 5 \\ 0 & 1 & 0 \\ -5 & 9 & -7 \end{vmatrix} = 15 \begin{vmatrix} 7 & 5 \\ -5 & -7 \end{vmatrix}$$
$$= 15(-49 + 25) = -15 \times 24 = -360.$$

The foregoing determinant is a *magic square*, and can be reduced in many different ways.

Trace the following changes in the magic square determinant of the fifth order:

$$\begin{vmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{vmatrix} = 65 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{vmatrix} = 65 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 23 & -18 & -16 & -9 & -7 \\ 4 & 2 & 9 & 16 & 18 \\ 10 & 2 & 9 & 11 & -7 \\ 11 & 7 & 14 & -9 & -2 \end{vmatrix}$$

$$= -65 \begin{vmatrix} 18 & 16 & 9 & 7 \\ 2 & 9 & 16 & 18 \\ 2 & 9 & 11 & -7 \\ 7 & 14 & -9 & -2 \end{vmatrix} = -65 \begin{vmatrix} 0 & -65 & -90 & 70 \\ 0 & 0 & 5 & 25 \\ 2 & 9 & 11 & -7 \\ 1 & -13 & -42 & 19 \end{vmatrix} = -65 \begin{vmatrix} 0 & -65 & -90 & 70 \\ 0 & 0 & 5 & 25 \\ 0 & 35 & 95 & -45 \\ 1 & -13 & -42 & 19 \end{vmatrix}$$

$$= 125 \times 65 \begin{vmatrix} -13 & -18 & 14 \\ 0 & 1 & 5 \\ 7 & 19 & -9 \end{vmatrix} = 125 \times 65 \begin{vmatrix} 1 & 20 & -4 \\ 0 & 1 & 5 \\ 7 & 19 & -9 \end{vmatrix} = 125 \times 65 \begin{vmatrix} 1 & 20 & -4 \\ 0 & 1 & 5 \\ 0 & -121 & 19 \end{vmatrix}$$

$$= 125 \times 65 \begin{vmatrix} 1 & 5 \\ -121 & 19 \end{vmatrix} = 125 \times 65(19 + 605) = 125 \times 65 \times 624 = 5070000.$$

The pupil may find *amusement*, as well as practice, in reducing this determinant in a great many different ways.

656. By applying the principles now established, we can readily solve groups of simultaneous linear equations containing two, three, or more unknown numbers.

Take, for example, the equations solved in § 647

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$

$$a_2x + b_2y + c_2z = d_2, \tag{2}$$

$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

If we multiply (1) by $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, (2) by $-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, and (3) by $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, and add the results, we have

$$\left(a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right) x$$

$$+\left(b_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right) y$$

$$+\left(c_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + c_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right) z$$

$$= d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - d_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + d_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

But by Property 6 this may be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x + \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} y + \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} z = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

Now, by Property 3 the coefficients of y and z vanish; and hence

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

657. In like manner the values of y and z are obtained. In finding these values, instead of rearranging the terms, we change the signs of the determinants for y by interchanging the columns containing the d's and b's; and for z, by interchanging the columns containing the d's and c's.

Similarly, the values of the unknown numbers in a system of linear equations with any number of unknown numbers may be written down in determinant form.

CHAPTER XXIII.

CURVE TRACING.

658. The position of any place on the earth's surface is defined if its latitude and longitude are given, the latitude being measured north and south from the equator, and the longitude east and west from any convenient meridian. North latitudes are defined as *positive*, and south latitudes as *negative;* east longitudes as *positive*, and west longitudes as *negative* (§ 60).

659. Similarly, we can locate the position of a point in a plane if we know its *distances* and *directions* from two fixed intersecting straight lines. Let these lines be drawn perpendicular to each other, as XX' and YY', and let them intersect at O.

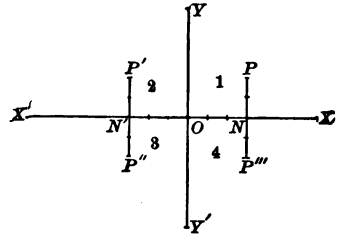


FIG. 1.

The line XX' is called the *axis of x* or the *axis of abscissas;* the line YY', the *axis of y* or the *axis of coördinates;* and the two together, the **axes of coördinates.** The point O is called the **origin.** Each of the four portions into which the two lines divide the plane of the paper is called a **quadrant,** and the four quadrants are numbered as indicated in the figure.

660. Distances measured along OX, or parallel to OX, from YY' *to the right,* are defined as *positive;* along OX', or parallel to OX', from YY' *to the left,* as *negative.* Distances measured

along OY, or parallel to OY, from XX' *upward*, are defined as *positive;* along OY', or parallel to OY', *downward*, as *negative.* Thus, if the point P is 3 units of length from YY' in the positive direction, and 2 units of length from XX' in the positive direction (some convenient unit of length being taken), its position is completely determined; for from O we have only to measure along OX a distance ON equal to 3, and then from N measure upward and parallel to OY a distance NP equal to 2, and we arrive at *the position of the point P.*

661. The two distances ON and NP are called the **coördinates** of the point P. Distances along the axis of x are usually designated by the letter x, and distances along the axis of y by the letter y. Then, in the example given for the coördinates of P, we have $x = 3$, and $y = 2$, usually written (3, 2), or in general (x, y). The point $(-3, 2)$ is found in the second quadrant by measuring from O along OX' a distance ON' equal to -3, and then from N' measuring upward and parallel to OY a distance $N'P'$ equal to 2, and we arrive at the position of the point P'.

Similarly the point P'', whose coördinates are $(-3, -2)$, is found in the third quadrant; and the point P''', whose coördinates are $(3, -2)$, is found in the fourth quadrant.

662. It is thus seen, that, to determine the position of a point in a plane, it is not sufficient to know simply the distances of the point from the two lines XX' and YY', but the *directions* must also be known. If the distances alone were known, four points, one in each quadrant, would satisfy the conditions; but, when the directions also are assigned, the position of the point is completely determined.

EXERCISES.

1. What are the coördinates of the origin?

2. In what quadrants are the following points: $(2, -6)$, $(-3, 5)$, $(-4, -1)$, (a, b)?

Draw the axes of coördinates and locate these points, using $\frac{1}{4}$ of an inch as a unit of measure; and use the same unit of measure in each of the following examples.

3. Locate the following points: (7, 8), (1, 0), (0, 3), (0, 0), (5, −4), (−1, −7), and (−5, 0).

4. Locate the following points: (−3, 2), (−3, −2), (3, −2), (0, 3), (−1, 2), (1, $2\frac{1}{2}$), ($\frac{1}{2}$, −2).

5. Draw the triangle which has for vertices the following points: (1, 1), (−3, 2), (0, 0).

6. Draw the quadrilateral having for its vertices the following points: (2, −3), (−3, 4), (3, −4), (−2, 3).

7. Draw the polygon having for its vertices the following points: (2, 0), (1, 1), (−1, 1), (0, −2), (−1, −1), (1, −1).

The student is recommended either to provide himself with paper ruled in small squares, or to draw on the paper two straight lines perpendicular to each other, and lay off distances from the origin along these lines equal to $\frac{1}{4}$ of an inch, and draw lines parallel to the original lines from these points, thus dividing the paper into small squares.

GEOMETRICAL REPRESENTATION OF EQUATIONS WITH TWO UNKNOWN QUANTITIES, X AND Y.

I. Equations of the first degree in x and y.

663. Let $2x + y - 1 = 0$ be an equation of the first degree in x and y. For every value of x we have a corresponding value of y; so that, if x varies continuously, y also varies continuously.

Let a number of corresponding values of x and y be found. Thus,

(1) If $x = 0$, $y = 1$.	(2) If $x = -1$, $y = 3$.
If $x = 1$, $y = -1$.	If $x = -2$, $y = 5$.
If $x = 2$, $y = -3$.	If $x = -3$, $y = 7$.
If $x = 3$, $y = -5$.	If $x = -4$, $y = 9$.
If $x = 4$, $y = -7$.	If $x = -5$, $y = 11$.

These values of x and y form the coördinates of a set of points which can readily be located. When these points are connected by a line, we have a geometrical representation or picture of the equation. This line is called the **locus of the equation**; and hence the locus of an equation may be defined as *the line the coördinates of every point of which satisfy the given equation.*

If all possible values of x should be substituted in the equation, and the corresponding values of y found, the sets of values would represent the coördinates of an infinite number of points; and these points, if located exactly with reference to the axes of coördinates, would form a *continuous* line or locus. In practice, only those values of x are taken which are most convenient, usually integral, and the coördinates of a set of these points found.

664. To construct the locus of the equation $2x + y - 1 = 0$, let us take the first four positive values of x given in § 663.

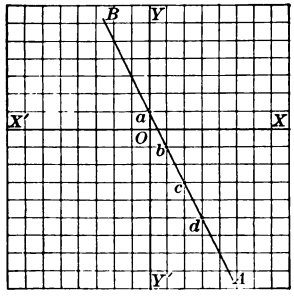


FIG. 2.

Draw the axes of coördination, and assuming some convenient unit of measure, say $\frac{1}{4}$ of an inch, mark the paper into squares, as in the figure.

The point (0, 1) is found at a, a distance of 1 unit above O. The point $(1, -1)$ is found by measuring a distance equal to 1 on OX, and then a distance equal to 1 *below* OX, parallel to OY, the point b. The point $(2, -3)$ is at c, whose distance from OY is 2, and from OX is -3. The point $(3, -5)$ is at d, whose distance from OY is 3, and from OX is -5. Through the points a, b, c, and d draw the indefinite line AB, and we have the geometrical representation or locus of the equation $2x + y - 1 = 0$.

The foregoing process might be continued indefinitely for both positive and negative values of x. Fractional values might also be used, and points located between those found above.

EXERCISES.

1. Construct the locus of $2x + y = 6$.

SUGGESTION. Compute a table of corresponding values of x and y, and, with these sets of corresponding values as coördinates of points, proceed as above. The line AB in Fig. 3 is the locus of the equation.

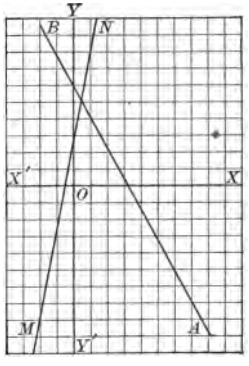


FIG. 3.

2. Construct the locus of $y = 5x + 3$.

The line MN in Fig. 3 is the required locus.

3. Construct the locus of

$$3x + 5y = 15.$$

Let x and y in turn equal 0, giving $x = 5$, and $y = 3$.

Locate the points (0, 3) and (5, 0), and through these points draw an indefinite line.

Construct the loci of the following equations:

4. $2x + y = 6$.
5. $7x - 4y = 0$.
6. $x + 5y = 5$.
7. $y = 2x + 2$.
8. $x - y = 3$.
9. $x - y = 0$.
10. $y = -4x + 6$.
11. $2x - 1 = y$.
12. $\frac{y - 6x}{2} = 1$.
13. $\frac{x}{2} + \frac{y}{3} = 1$.

665. Each of the above loci will be found to be a straight line, and it may be assumed that *every equation of the first degree in* x *and* y *is represented by a straight line;* and, since two straight lines can intersect in but *one* point, the point of intersection of two straight lines is *the geometrical solution of the two equations of these lines.*

Let us, for example, find the geometrical solution of the equations $\begin{cases} x + y + 1 = 0, \\ x + 2y + 4 = 0. \end{cases}$

Compute, as before, sets of values of x and y for each equation, and construct loci, as in Fig. 4.

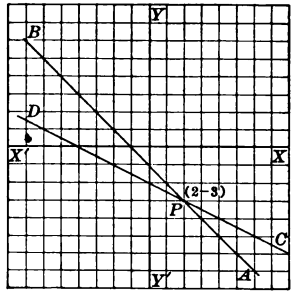


Fig. 4.

The lines will be found to intersect in the point P, whose coördinates are $x = 2$, and $y = -3$.

These values of x and y must satisfy both equations: for, since every point in AB must satisfy $x+y+1=0$, the coördinates of P must satisfy this equation. Likewise, since the coördinate of every point in CD must satisfy

$$x + 2y + 4 = 0,$$

the coördinates of P must satisfy this equation. Hence $x=2$, and $y=-3$, satisfy both equations simultaneously, and represent the solution of the equations.

14. Find the coördinates of the point of intersection of the loci given in Fig. 3.

Solve geometrically the following sets of equations:

15. $\begin{cases} x+y=6, \\ 2x-y=9. \end{cases}$

16. $\begin{cases} x+y=4, \\ x-y=0. \end{cases}$

17. $\begin{cases} x+2y=13, \\ 3x+y=14. \end{cases}$

18. $\begin{cases} 2x+3y=2, \\ 6x+3y=4. \end{cases}$

19. $\begin{cases} 3x+4y=-1, \\ 9x-2y=4. \end{cases}$

20. $\begin{cases} 4x+5y=22, \\ 5x-2y=11. \end{cases}$

II. Equations of the second degree in x and y.

666. The points whose coördinates satisfy the equation $y = x^2 - 2x + 2$ do not lie in a straight line. They do, however, lie in some line which is determined by the equation. To construct this line it is necessary to find a number of points, and then draw a continuous curve through them. Thus,

For $x = 0$, $y = 2$.
For $x = 1$, $y = 1$.
For $x = 2$, $y = 2$.
For $x = 2\frac{1}{2}$, $y = 3\frac{1}{4}$.
For $x = 3$, $y = 5$.
For $x = 3\frac{1}{2}$, $y = 7\frac{1}{4}$.
For $x = -\frac{1}{2}$, $y = 3\frac{3}{4}$.
For $x = -1$, $y = 5$.

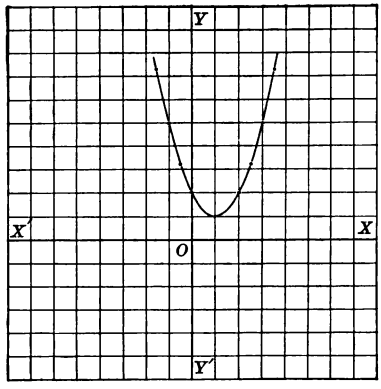


FIG. 5.

The points tabulated above are plotted in Fig. 5, and a continuous curve drawn through them. This curve is the locus of the equation $y = x^2 - 2x + 2$.

21. Construct the loci of $y^2 = 4x + 2$ (Fig. 6), and $xy = 4$ (Fig. 7).

Writing the second equation in the form $y = \frac{4}{x}$, and assigning both positive and negative values to x, we get the following table of values for x and y.

For $x = 0$, $y = \infty$.
For $x = \pm 1$, $y = \pm 4$.
For $x = \pm 2$, $y = \pm 2$.
For $x = \pm 3$, $y = \pm \frac{4}{3}$.
For $x = \pm 4$, $y = \pm 1$.
. . . . . .
For $x = 0$, $y = \infty$.

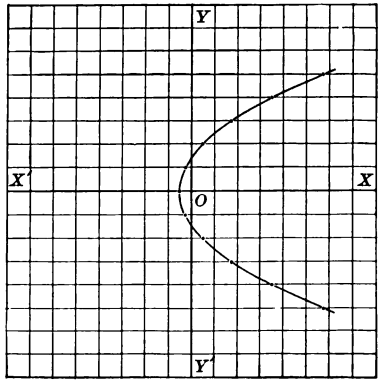


FIG. 6.

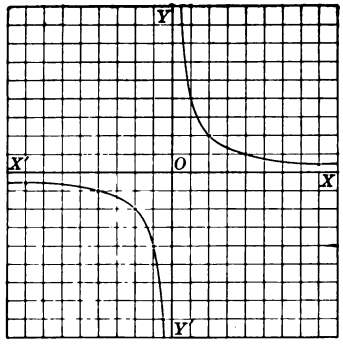


FIG. 7.

Plotting the points that correspond to the positive values of x and y, we have a branch of the curve in the first quadrant.

Corresponding to the negative values, we have a branch of the curve in the third quadrant.

It is thus seen that in this case the curve is composed of two branches, as indicated in Fig. 7.

Construct the following loci:

22. $4x = y^2$.
23. $3x^2 - 4y^2 = 12$.
24. $3x^2 + 4y^2 = 12$.
25. $x^2 + y^2 = 4$.
26. $x^2 = 4y$.
27. $x + y + x^2 = -1$.

GEOMETRICAL REPRESENTATION OF THE ROOTS OF AN EQUATION.

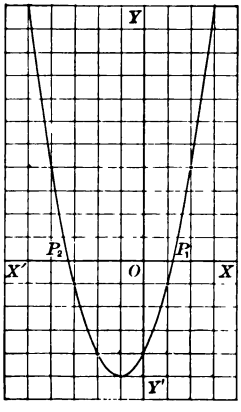


FIG. 8.

667. Let $y = x^2 + 2x - 4$. Compute a table of values of x and y and construct the locus (Fig. 8). At the points P_1 and P_2, where the curve cuts the axis of x, we have $y=0$, and hence $x^2 + 2x - 4 = 0$. The values of x for these points are OP_1 and OP_2; and as these are the values of x which make the expression $x^2 + 2x - 4 = 0$, they represent the values of the roots of the equation. These are readily seen to be $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$.

668. If the value of the absolute term be increased, the curve will be

moved upward, and the points P_1 and P_2 will approach each other. When we have $y = x^2 + 2x + 1$, the points P_1 and P_2 coincide, and the curve simply touches the axis of x, or, we might say, it cuts the axis of x in coincident points. For these points we have $x^2 + 2x + 1 = 0$, the roots of which are readily seen to be each equal to unity. This is the geometrical representation of the condition of equal roots. Thus, when the curve touches the axis of x, but does not cross it, the two values of x are equal.

669. If the value of the absolute term be still further increased, the curve is moved upward still farther, and does not cut the axis of x at all, or it is said to cut the axis in *imaginary points*. In this case the roots are imaginary. Thus, if in $y = x^2 + 2x + 2$, we put $y = 0$, and find the corresponding values of x, we get $x = -1 + \sqrt{-1}$ and $-1 - \sqrt{-1}$. These values of x are imaginary, and the corresponding points cannot be located by means of the system of coördinates we are using. Hence, when the locus does not cut the axis of x, the roots of the equation resulting by putting $y = 0$ are imaginary.

Construct the loci of the following equations, and determine the values of x which make $y = 0$:

1. $y = x^3 - 1$.
2. $y = x^3 + 1$.
3. $4y = \frac{x^3}{3} - 2x^2 + 16$.
4. $y^{\frac{1}{2}} = 2 - x^{\frac{1}{2}}$.
5. $4y = x^3$.
6. $y = x^3 - 2x$.
7. $y = 2x^2 + 1$.
8. $y = \pm 2\sqrt{x + 4}$.
9. $y^2 = 16 - x^2$.
10. $y = x^3 - 6x^2 + 11x - 6$.

CHAPTER XXIV.

PERMUTATIONS AND COMBINATIONS.

PERMUTATIONS.

670. The different orders in which a set of things can be arranged are called **permutations.**

Thus, the permutations of the three letters *a, b, c,* taken two at a time, are

ab, ba, ac, ca, bc, cb.

In like manner the three digits 1, 2, 3, taken two at a time, express the six numbers

12, 21, 13, 31, 23, 32.

671. The number of permutations of n things taken m at a time is equal *to the continued product of the* m *successive integers from* n *to* n − (m − 1).

For suppose the n things to be n letters, $a, b, c, d, \cdots$, and denote by ${}^{n}p_1, {}^{n}p_2, {}^{n}p_3, \cdots {}^{n}p_n$, the number of permutations that can be formed from n things taken 1, 2, 3, $\cdots m$ at a time. The number of permutations of n letters taken one at a time is evidently equal to the number of letters; that is,

$$ {}^{n}p_1 = n. $$

If we put each of the n letters before each of the remaining $n-1$ letters, we obtain

$$ {}^{n}p_2 = {}^{n}p_1 \times (n-1) = n(n-1). $$

Again, if we put each of these ${}^{n}p_2$ permutations before each of the remaining $n-2$ letters, we obtain

$$ {}^{n}p_3 = {}^{n}p_2 \times (n-2) = n(n-1)(n-2). $$

Again, if we put each of these ${}^{n}p_3$ permutations before each of the remaining $n-3$ letters, we obtain

$${}^{n}p_4 = {}^{n}p_3 \times (n-3) = n(n-1)(n-2)(n-3).$$

We may proceed in this way until we obtain ${}^{n}p_{m-1}$; and, if we put each of these ${}^{n}p_{m-1}$ permutations before each of the remaining $n-(m-1)$ letters, we shall obtain

$${}^{n}p_m = {}^{n}p_{m-1} \times [n-(m-1)] = n(n-1)(n-2)(n-3)\cdots[n-(m-1)].$$

672. If all of the n letters are to be arranged together, then m in the above formula becomes equal to n, and $n-(m-1) = n-n+1 = 1$, and

$${}^{n}p_n = n(n-1)(n-2)\cdots 3, 2, 1.$$

Hence *the number of permutations of* n *things taken all together is equal to the continued product of all the* n *integers from* n *down to* 1 *or from* 1 *to* n.

Thus the number of permutations of four digits taken together is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

673. The continued product of the successive numbers from n to 1 or from 1 to n is denoted by the symbol $\lfloor n$, which is read "factorial n." Thus $\lfloor 5$ denotes $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

The symbol $n\,!$ is also used to express factorial n.

674. In the above formula for ${}^{n}p_n$, all the letters were supposed to be different. But if we suppose p of them, for example, to be the same, then the $\lfloor p$ permutations, arising from the presence of these letters when different, reduce to 1. If we suppose another group of q letters to be the same, then $\lfloor q$ permutations, in like manner, reduce to 1.

Hence, to find the number of permutations of n things taken all together, one set of p of them being alike and another set of q of them being alike, we must divide ${}^{n}p_n$ by $\lfloor p$ and $\lfloor q$; and the required number of permutations will then be expressed by $\dfrac{{}^{n}p_n}{\lfloor p \lfloor q}$; and similarly if there were three or more sets of like things.

EXERCISES.

1. How many numbers can be expressed by 9 digits taken 4 at a time?

Here $n = 9$, and $m = 4$, and hence

$$^9p_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3024.$$

The subscript 4 shows how many factors are to be taken, the superscript 9 being the first.

2. In how many different orders can 6 boys sit on a bench?

$$^6p_6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

3. How many permutations can be formed of the letters in the word *possessions*, taken all together?

Here $n = 11$, and the letter s occurs 5 times, and o twice.

Hence $\frac{^np_n}{\lfloor 5 \lfloor 2} = \frac{\lfloor 11}{\lfloor 5 \lfloor 2} = \frac{11 \cdot 10 \cdot 9 \cdots 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 166320.$

4. In how many ways can the letters a, b, c, d, e, be arranged, taken 3 at a time? Taken 4 at a time?

5. In how many ways could the seven prismatic colors be arranged, taken all together?

6. How many signals could be made with 5 flags of different colors, taken 1, 2, 3, 4, and 5 at a time?

7. In how many ways could a party of 5 persons be seated at a table?

8. How many permutations could be formed from the letters of the word *Columbia*, taken all together? Taken 5 at a time?

9. How many signals in all could be made with 7 flags of different colors?

10. How many permutations can be formed from the letters of the word *consonant*, taken together? From the letters of the word *Mississippi*?

COMBINATIONS.

675. The different sets that can be formed from a given number of things without regard to the order in which they occur, are called **combinations.**

Thus, the possible combinations of the letters *a, b, c,* taken two at a time, are *ab, ac, bc*; since *ab* and *ba,* though different permutations, are the same combination, as are also *ac, ca,* and *bc, cb.*

676. Any combination of m things will produce $\lfloor\underline{m}$ permutations, for the set of m things that form a given combination can be permuted in $\lfloor\underline{m}$ different ways.

677. The number of combinations of n things taken m at a time is equal *to the number of permutations of* n *things taken* m *at a time, divided by* $\lfloor$m.

For if we denote the number of possible combinations of n things taken m at a time by ${}^nC_m$, then, by § 674,

$$ {}^nC_m \times \lfloor\underline{m} = {}^nP_m.$$

$$\therefore {}^nC_m = {}^nP_m \div \lfloor\underline{m} = \frac{n(n-1)(n-2)\cdots[n-(m-1)]}{\lfloor\underline{m}}.$$

678. Since for every combination of m things which we take out of n things we leave a combination of $n - m$ things, it follows that *the number of combinations of* n *things taken* m *at a time is equal to the number of combinations taken* n − m *at a time; that is,* ${}^nC_m = {}^nC_{n-m}$.

EXERCISES.

1. How many combinations can be formed with the letters *a, b, c, d, e,* taken 3 at a time?

Here $n = 5$, and $m = 3$; and hence

$${}^5C_3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10.$$

The number of combinations of 5 letters taken 2 at a time is also 10, since $\frac{5 \times 4}{1 \times 2} = 10$.

2. How many different pickets of 5 men and an officer can be made from a squad of 20 men and 3 officers?

Here $n = 20$, and $m = 5$; and hence

$$^{20}C_5 = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 15504.$$

Since one of 3 officers may go with each picket, the whole number of pickets is $15504 \times 3 = 46512$.

3. How many combinations can be formed of the letters of the word *longitude*, taken 4 at a time? How many combinations taken 5 at a time?

4. How many sums can be formed of the digital numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, taken 3 at a time? How many taken 6 at a time?

5. A number is the product of 5 different prime factors, including unity. How many divisors has the number?

6. How many different selections of 4 pieces each can be made from 12 different pieces of money?

7. How many different sums of money can be formed of a cent, a three-cent piece, a half dime, a dime, a quarter, and a half dollar?

8. In a school of 18 boys and 15 girls, how many classes could be formed, each to consist of 4 boys and 3 girls?

9. Out of the 7 prismatic colors how many combinations of 3 colors each can be made?

10. Company A contains one man more than Company B, and the number of combinations of 3 men each that can be made from Company A is to the number of similar combinations that can be made from Company B as 21 to 20. How many men in each company?

APPENDIX.

679. The following supplementary exercises are designed for advanced classes, and especially for students who are preparing to enter higher institutions which may have exceptional entrance requirements in algebra. Many of the problems have been selected from recent college examination papers; and each series contains as difficult problems as any found in the papers examined.

680. Complex Fractions. (§ 273.)

Simplify

1. $\dfrac{\dfrac{1-x^3}{1+x^3}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{1+x}{1-x}}$.

2. $\dfrac{\dfrac{1-x^2}{1+y}\left(\dfrac{x}{1+x}-1\right)}{1-\left(\dfrac{1}{1-y}-\dfrac{x^2+y^2-x+y}{1-y^2}\right)}$.

3. $\dfrac{1-\dfrac{2+\dfrac{1}{x}}{2-\dfrac{1}{x}}}{1-\dfrac{x+\dfrac{1}{2}}{x-\dfrac{1}{2}}}$.

4. $\left\{\dfrac{\dfrac{x^6+1}{x^2+1}}{\dfrac{x^3+1}{x+1}}\right\}[1+x(x-1)]$.

5. $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}} \div \dfrac{1}{a+\dfrac{1}{b}}$.

6. $\dfrac{\dfrac{x^2}{y}+\dfrac{y^2}{x}}{x+\dfrac{y^2}{x}-y} \div \left(1+\dfrac{x}{y}\right)$.

7. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}}$.

8. $\dfrac{\left(a^2+\dfrac{b^4}{a^2+b^2}\right)(a^2-b^2)}{1-\dfrac{b}{a+b}-\frac{1}{2}\left(1-\dfrac{a-b}{a+b}\right)}$.

9. $\dfrac{\dfrac{1}{x}-\dfrac{x+y}{x^2+y^2}}{\dfrac{1}{y}-\dfrac{y+x}{y^2+x^2}}+\dfrac{\dfrac{1}{x}-\dfrac{x-y}{x^2+y^2}}{\dfrac{1}{y}-\dfrac{y-x}{y^2+x^2}}$.

10. $\dfrac{\dfrac{(x^3-y^3)(x+y)}{(x^3+y^3)(x-y)}}{\dfrac{x^4-y^4}{(x^3-y^3)(x+y)}}\times\dfrac{x^2-xy+y^2}{x^2+xy+y^2}$.

11. $\dfrac{x^2+(b+c)x+bc}{x^2+(a-d)x-ad}\times\dfrac{x^2-d^2}{x^2-c^2}\Big/\dfrac{x^2+(b+d)x+bd}{x^2+(a-c)x-ac}$.

681. Rationalizing the Divisor. (§ 398.)

1. $\dfrac{\sqrt{10}}{\sqrt{5}+\sqrt{2}-\sqrt{6}}$.

2. $\dfrac{4+\sqrt{2}-\sqrt{3}}{3-\sqrt{2}+\sqrt{3}}$.

3. $\dfrac{1+\sqrt{3}+\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}$.

4. $\dfrac{\sqrt{(3+2\sqrt{2})}-\sqrt{2}}{\sqrt{(3-2\sqrt{2})}+\sqrt{2}}$.

5. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{2}-\sqrt{32}+\sqrt{50}}$.

6. $\dfrac{3+2\sqrt{2}-3\sqrt{3}-2\sqrt{6}}{1+\sqrt{2}-\sqrt{3}-\sqrt{6}}$.

7. $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}$.

8. $\dfrac{1}{1+\sqrt{-2}+\sqrt{-3}}$.

9. $\dfrac{\sqrt{3}}{2-\sqrt{3}}-\dfrac{2-\sqrt{-2}}{2+\sqrt{-2}}$.

10. $\dfrac{1}{\sqrt[3]{25}-\sqrt[3]{20}+2\sqrt[3]{2}}$.

SUGGESTION. — Multiply both terms by $\sqrt[3]{5}+\sqrt[3]{4}$.

11. $\dfrac{1}{\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}}$.

12. $\left(\sqrt{\dfrac{a+b}{a-b}}-\sqrt{\dfrac{a-b}{a+b}}\right)\sqrt{\dfrac{a^3-b^3}{(a+b)^2-ab}}$.

682. Equations involving Radicals. (§ 410.)

Solve

1. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}$.

2. $\dfrac{1}{\sqrt{x-1}}+\dfrac{1}{\sqrt{x+1}}=\dfrac{1}{\sqrt{x^2-1}}$.

3. $\sqrt{a-x}+\sqrt{b-x}=\dfrac{b}{\sqrt{b-x}}$.

4. $\sqrt{2+x}+\sqrt{2+x+\sqrt{2-x}}=\sqrt{2-x}$.

5. $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\dfrac{4}{3}$.

6. $\dfrac{\sqrt{5x}-2}{\sqrt{5x}+2}=\dfrac{2\sqrt{5x}-7}{3\sqrt{5x}-8}$.

SUGGESTION. — Equations 5 to 10 may be solved by proportion.

7. $\dfrac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\dfrac{1}{a}$.

8. $\dfrac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}=\dfrac{a+\sqrt{b^2-a^2}}{a-\sqrt{b^2-a^2}}$.

9. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\dfrac{a+b}{a-b}$.

10. $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=17$.

11. $\sqrt[3]{14-x}+\sqrt[3]{14+x}=4$.

12. $\begin{cases}\sqrt{2x-y}=\sqrt{x-y}+1,\\ x^2+4y=17.\end{cases}$

13. $\begin{cases}\dfrac{7}{\sqrt{x}}+\dfrac{4}{\sqrt{y}}=40,\\ \dfrac{1}{\sqrt{x}}+\dfrac{2}{\sqrt{y}}=4.\end{cases}$

14. $x^{\frac{1}{n}}-6x^{-\frac{1}{n}}=1$.

683. Imaginary Numbers. (§ 420.) Simplify

1. $(6-5\sqrt{-1})(5+3\sqrt{-1})$.
2. $(2+\sqrt{-6})(3+2\sqrt{-10})$.
3. $\sqrt{-10}\div-\sqrt{5}$.
4. $\sqrt{15}\div\sqrt{-3}$.
5. $-8\sqrt{26}\div 2\sqrt{-2}$.
6. $(3-\sqrt{-2})^3$.
7. $(3-\sqrt{-5})^{-2}$.
8. $(3-\sqrt{-1})\div(2+\sqrt{-1})$.
9. $(1+\sqrt{-2}+\sqrt{-3})(1+\sqrt{-2}-\sqrt{-3})$.
10. $(9\sqrt{-3}-11\sqrt{-2})\div(\sqrt{-3}+\sqrt{-2})$.
11. $(-2\sqrt{6}-\sqrt{-1})\div(\sqrt{-2}-\sqrt{2}+\sqrt{-3})$.
12. $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}-\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}$.
13. $\sqrt{34-12\sqrt{-2}}$. (§ 407.)
14. $\sqrt{40-42\sqrt{-1}}$.
15. $\sqrt{-5+12\sqrt{-1}}$.
16. $\sqrt{-26+6\sqrt{-3}}$.

684. Fractional and Negative Exponents. (§ 436.)

Extract the square root of

1. $x^3y^{-\frac{2}{3}}-4x^{\frac{3}{2}}y^{-\frac{1}{3}}+6-4x^{-\frac{3}{2}}y^{\frac{1}{3}}+x^{-3}y^{\frac{2}{3}}$.
2. $a^2-a^{\frac{3}{2}}\sqrt{x}+2a^{\frac{1}{2}}\sqrt{x^3}+4x(x-\frac{1}{16}a)$.

Extract the cube root of

3. $8x^{-2}-12x^{-\frac{11}{6}}+6x^{-\frac{5}{3}}-x^{-\frac{3}{2}}$.
4. $a^{\frac{1}{2}}-9a^{\frac{2}{3}}+33a^{\frac{5}{6}}-63a+66a^{\frac{7}{6}}-36a^{\frac{4}{3}}+8a^{\frac{3}{2}}$.

Resolve into factors

5. $x - y$.
7. $a^{-1} - b$.
9. $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + \frac{1}{4}y$.
6. $x^{4n+1} - x^{-2}$.
8. $x^{\frac{2}{5}} - y^{\frac{2}{5}}$.
10. $x^{-\frac{3}{2}} - 2x^{-\frac{3}{4}}y^{\frac{1}{2}} + y$.

Simplify

11. $\dfrac{a^0 a^{\frac{1}{2}}}{a^{\frac{1}{3}}} \times a^{\frac{2}{3}}a^{-\frac{1}{2}}$.
12. $\left(\dfrac{a^{x+y}}{a^y}\right)^x \div \left(\dfrac{a^y}{a^{y-x}}\right)^{x-y}$.
13. $[(x^{\sqrt{-a}}y^{\sqrt{b}})^{\sqrt{-ab}}]^b_a$.
14. $\dfrac{a^{-2}b^{\ 2}}{a^{-2}+b^{\ 2}} \times \dfrac{a^2+b^2}{a^{-1}-b^{-1}}$.
15. $\left(a^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{ax^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.
16. $\dfrac{9^{n+\frac{1}{4}} \times \sqrt[3]{3+3^n}}{3\sqrt{3^{-n}}}$.
17. $\dfrac{9^{-2} \div 16^{\frac{7}{4}}}{81^{-\frac{3}{2}} \div 4^{\frac{3}{2}}}$.
18. $\dfrac{a^n b^{n-1}c^{-n}}{x^{-3}y} \times \dfrac{a^{-n}bc^{n-1}}{x^2y^{-2}}$.
19. $a^{\frac{2}{3}}b^{-\frac{4}{3}} \times a^{-\frac{1}{2}}b^{-\frac{2}{3}} \times a^{\frac{3}{2}}b^{\frac{5}{2}}$.
20. $(a^{\frac{3}{2}}b^{-\frac{2}{3}})^{\frac{2}{3}} \times (a^{-3}b^{-\frac{5}{3}})^{\frac{1}{3}}$.
21. $x^{2a}y^{-\frac{a}{2}} \times x^{-\frac{3a}{2}}z^{-b} \times (y^{\frac{3a}{2}}z^{3b})^{\frac{1}{2}}$.
22. $[(x^{\frac{a}{3}}y^{-\frac{b}{2}}z^{ab})^{\frac{3a}{2}}]^{-2b}$.
23. $(x^{(a-b)^2} \div x^{(a+b)^2}) \times x^{4ab}$.
24. $[(a^{m-n})^{m+n} \times (a^{-m})^{m-n}]^{\frac{1}{n}}$.
25. $(e^x + e^{-x})^2 - (e^x - e^{-x})^2$.
26. $\left(\dfrac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}\right)^2 + \left(\dfrac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}\right)^2$
27. $x^{a+2b}y^{3a-b} \times x^{a-b}y^{2a+b} \div x^{3a+b}y^{5a}$.
28. $\frac{1}{2}(x-1)^{\frac{1}{2}}(x+1)^{-\frac{1}{2}} + \frac{1}{2}(x+1)^{\frac{1}{2}}(x-1)^{-\frac{1}{2}}$.
29. $(ab^{-1}+1)[a - \frac{1}{2}(1+\sqrt{-3})b] \times [a - \frac{1}{2}(1-\sqrt{-3})b]$.

685. Quadratic Equations. (§§ 475–485.)

Solve the following groups of equations:

1. $\begin{cases} x^3 + y^3 = 189, \\ x + y + \sqrt{x+y} = 12. \end{cases}$
2. $\begin{cases} x^2y^8 + y^2 = 10, \\ xy^4 + y = 4. \end{cases}$
3. $\begin{cases} x^2 + xy + 4y^2 = 6, \\ 3x^2 + 8y^2 = 14. \end{cases}$
4. $\begin{cases} x^2 + y^2 - 15(x+y) = -64, \\ 2xy + 31(x+y) = 289. \end{cases}$
5. $\begin{cases} x^n + y^n = a, \\ xy = b. \end{cases}$
6. $\begin{cases} x^2 + y^2 = 17, \\ x + y = xy + 1. \end{cases}$

686. Equations of Higher Degrees. (§§ 471, 472.)

1. $3x^3 + 7x^2 + 3x + 2 = 0.$

SUGGESTION. — Divide by $x + 2$, and then solve the resulting quadratic.

2. $54x^3 - 39x^2 - 26x + 16 = 0.$
3. $x^4 + 6x^3 + 5x^2 - 12x - 12 = 0.$
4. $x^6 + 15ax^3 + 44a^2 = 0.$
5. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
6. $x^5 - x^4 - 64x^2 + 64x = 0.$
7. $x^4 - 2x^3 + x - 132 = 0.$

687. Proportion and Progression.

1. If $a:b = c:d$, prove $a:b = \sqrt{3a^2 + 5c^2} : \sqrt{3b^2 + 5d^2}$.

2. Find a mean proportional between (1) $4x^3 - 3x - 1$ and $x - 1$; (2) $2x - 3$ and $2x^3 + x^2 - 4x - 3$.

3. Two numbers whose difference is d, are to each other as a to b. What are the numbers?

4. Prove that in a geometrical progression $s = \dfrac{rl - a}{r - 1}$.

5. Show that if $a + b$, $2b$, $b + c$ are in harmonic progression, then a, b, c will be in geometrical progression.

6. The sum of the first six terms of a geometrical progression is 728, and the sum of the third and fourth terms is 72. Write the series.

688. Convergent Infinite Series. (§ 573.)

7. Show that in a decreasing geometrical series to infinity

$$s = \frac{a}{1 - r}.$$

Find the sum to infinity of

8. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots.$
9. $-8 - \frac{2}{5} - \frac{1}{50} - \cdots.$
10. $1, \ -\dfrac{1}{a^2}, \ +\dfrac{1}{a^4}, \ -\cdots.$
11. $1 + \left(1 - \dfrac{1}{a}\right) + \left(1 - \dfrac{1}{a}\right)^2 + \cdots.$
12. $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{2} + \cdots.$
13. $\sqrt{\frac{3}{2}}, \ \frac{1}{8}\sqrt{2}, \ \frac{1}{32}\sqrt{\frac{2}{3}}, \ \cdots.$

Find the value of the following recurring decimals:

14. $.\dot{2}\dot{4}$. 15. $.8\dot{5}\dot{1}$. 16. $5\dot{2}\dot{1}$. 17. $.\dot{0}3\dot{7}$. 18. $.12\dot{1}3\dot{5}$.

19. Show that $0 \div a = 0$; $a \div 0 = \infty$; and $0 \div 0 =$ any number.

689. Exponential Equations. (§ 613.)

Solve the following exponential equations:

1. $2^{3x+5} = 5^{x-3}$.
2. $3^{2x-1} = 2 \cdot 5^{x+1}$.
3. $7^{2x} = 2^{x+2} \cdot 3^{x}$.
4. $m^{ax+b} = n$.
5. $3^{1-x-y} = 4^{-y}$.
6. $2^{4x-1} = 3^{5y-z}$.

690. Undetermined Coefficients.

Resolve into partial fractions

1. $\dfrac{x^2 - x + 1}{(x^2+1)(x-1)^2}$.
2. $\dfrac{x^4 - 3x^3 - 3x^2 + 10}{(x-3)(x+1)^2}$.
3. $\dfrac{3x^2 + 3x + 18}{x^3 - 8}$.
4. $\dfrac{x^2 + 3x + 1}{(x+1)^3}$.
5. $\dfrac{1 + 2x}{x^2(x-1)(x+2)^3}$.
6. $\dfrac{2x^2 + x + 1}{2x^2 + x - 3}$.

Expand into series

7. $\dfrac{1 - x}{1 + x + x^2}$.
8. $\dfrac{1 + x + x^2}{x - x^2}$.
9. $\dfrac{1 + 2x^2 + x^4}{1 - x^2 - x^4}$.

691. Determinants.

Solve the following linear equations:

1. $\begin{cases} x - 2y = 3, \\ 2x + 5y = 5. \end{cases}$
2. $\begin{cases} 3x + 4y = 4, \\ 6x + 6y = 1. \end{cases}$
3. $\begin{cases} 6x - 5y = 7, \\ 7x - 2y = 12. \end{cases}$
4. $\begin{cases} 8x + \frac{1}{2}y = 26, \\ 4x - 5y = -8. \end{cases}$
5. $\begin{cases} 3x + 5y = a, \\ 4x + 2y = b. \end{cases}$
6. $\begin{cases} ax + by = c, \\ cx + dy = a. \end{cases}$

Expand the following determinants:

7. $\begin{vmatrix} x+y & x-y \\ x-y & x+y \end{vmatrix}$
8. $\begin{vmatrix} x+y & x^2+y^2 \\ x-y & x^2-y^2 \end{vmatrix}$
9. $\begin{vmatrix} x-y & x^2-xy+y^2 \\ x+y & x^2+xy+y^2 \end{vmatrix}$

ANSWERS.

NOTE. — When several examples are given under the same number, in most cases the answer to the *last* only is given. Answers which, if given, would destroy the utility of examples, are omitted.

Page 11. — **2.** Son, $14; father, $42. **3.** Vest, $4; coat, $20. **4.** Less, 7; greater, 35. **5.** Less, 9; greater, 27. **6.** Shorter, 5 yds.; longer, 25 yds.

Page 12. — **7.** B, 50 sheep; A, 100 sheep. **8.** Less, 18; greater, 72. **9.** 20. **11.** Father's age, 45 yrs.; son's, 15 yrs. **12.** 135. **13.** Saddle, $6; harness, $18.

Page 13. — **15.** Shorter, 45 ft.; longer, 75 ft. **16.** 50 and 70. **17.** 1st person's, $575; 2d person's, $1225. **18.** Chain, $20; watch, $65. **19.** 1st party, 158 votes; 2d party, 206 votes. **20.** Men, 47; women, 62; children, 218. **21.** A, $3000; B, $1500; C, $7500. **22.** 1st, 7 yds.; 2d, 21 yds.; 3d, 14 yds. **23.** Shorter, 15 ft.; longer, 30 ft. **24.** A's share, $1200; B's, $2400; C's, $4800. **25.** A's share, $4; B's, $2; C's, $12. **26.** First, $400; second, $600; third, $800.

Page 14. — **27.** A's age, 40 yrs.; B's, 20 yrs.; C's, 35 yrs. **28.** 15. **29.** One, $60; other, $80. **30.** B's age, 14 yrs.; A's, 56 yrs. **31.** Boys, 80; girls, 160. **32.** Buggy, $60; horse, $120. **33.** Mother's age, 45 yrs.; daughter's, 15 yrs. **34.** A's share, $85; B's, $40. **35.** 30 and 45. **36.** Smaller, 79 acres; larger, 121 acres. **37.** 186 sheep. **38.** Standing, 35 ft.; broken off, 55 ft. **39.** B, $2000; A, $3500.

Page 15. — **40.** Each son, $3675; widow, $14,700. **41.** Shorter, 24 in.; longer, 40 in. **42.** 18. **43.** 1st, 12; 2d, 24; 3d, 40. **44.** Father's age, 48 yrs.; son's, 24 yrs. **45.** Mother's age, 35 yrs.; daughter's, 15 yrs. **46.** A's age, 72 yrs.; B's, 36 yrs.; C's, 18 yrs. **47.** Father, $80; elder son, $40; younger, $20. **48.** 1st candidate, 137 votes; 2d, 97 votes; 3d, 72 votes.

Page 24. — **1.** 16. **2.** 18. **3.** 6. **4.** 31. **5.** 18. **6.** 25. **7.** 72. **8.** 20.

Page 25. — **9.** 9. **10.** 51. **11.** 31. **12.** 28. **13.** y. **14.** $2x - y$. **15.** $2y$. **16.** 4. **17.** $-a$. **18.** $ab + b$. **19.** $b - 4$. **20.** $4a + ab + 2b$. **21.** $2b$. **22.** $2a + 7b$. **23.** $-2b^2$. **24.** $-b^2$. **25.** $-2xy$. **26.** $xy - 2y^2$.

Page 26. — **1.** 10. **2.** 26. **3.** 7. **4.** 24. **5.** 51.

Page 27. — **6.** $6a$. **7.** $6x$. **8.** $54x - 18$. **9.** $5xy - 13$.

Page 29. — **1.** 23. **2.** 55. **3.** 4. **4.** 45. **5.** 21. **6.** 4. **7.** 38. **8.** 0. **9.** -1. **10.** 88. **11.** 37. **12.** 14. **13.** 62. **14.** 18. **15.** 28.

16. 35. **17.** 92. **18.** 9. **19.** 7. **20.** 22. **21.** 11. **22.** 169. **23.** 28. **24.** 188. **25.** 7.

Page 39. — **1.** $x = 3$. **2.** $x = 3$. **3.** $x = 6$. **4.** $x = 8$. **5.** $x = 8$. **6.** $x = 7$. **7.** $x = 4$. **8.** $x = 2$. **9.** $x = 6$. **10.** $x = 3$.

Page 40. — **11.** $x = 2$. **12.** $x = 7$. **13.** $x = 6$. **14.** $x = 5$. **15.** $x = 11$. **16.** $x = 7$. **17.** $x = 3$. **18.** $x = 10$. **19.** $x = 10$. **20.** $x = 3$. **21.** Corn, 120 bu.; wheat, 240 bu.; oats, 360 bu. **22.** 1st, \$14; 2d, \$28; 3d, \$84. **23.** Quarters, 4; dimes, 16; nickels, 32. **24.** 1st year, \$900; 2d year, \$1800; 3d year, \$3600. **25.** B's, \$2000; A's, \$4000; C's, \$6000. **26.** 9 and 27. **27.** A's, \$28; B's, \$7. **28.** Younger, 8 yrs.; elder, 24 yrs.; father, 48 yrs.

Page 41. — **30.** In quarters, \$3; in dimes, \$1.20; in nickels, \$.60. **31.** Younger, \$18; elder, \$36; father, \$108. **32.** A, 30 yrs.; B, 40 yrs.; C, 50 yrs. **33.** 1st farm, 100 acres; 2d farm, 80 acres. **34.** Each sister, 170 acres; brother, 300 acres. **35.** A, \$50; B, \$130. **36.** B, \$2420; A, \$4840; C, \$6760. **37.** 9 hours; one travels 27 miles, other 36 miles.

Page 42. — **38.** First, 50; second, 80; third, 65. **39.** Serge, 75¢ per yard; silk, \$1.50. **40.** Oranges, 4 doz.; lemons, 8 doz.; pears, 16 doz. **41.** Vest, \$5; trousers, \$10; coat, \$15. **42.** Bonnet, \$5; dress, \$13; cloak, \$17. **43.** First, \$25; second, \$40; third, \$35. **44.** \$1. **45.** B, 40 acres; A, 60 acres; C, 60 acres. **46.** 9 hours.

Page 44. — **17.** $-15(a + b)$. **18.** $13xy(a - b)$. **19.** $32\sqrt{ab}$. **20.** $-17mn\sqrt{x}$.

Page 45. — **27.** $5(a + c)$. **28.** 0. **29.** $\sqrt{x}$. **30.** $5a\sqrt{xy}$.

Page 46. — **38.** $-4a^2b + 15ab + 2ab^2$. **39.** $5ax + 2ax^2 - x^2y^2$. **40.** $6x^2y$. **41.** $x^3 + 3x^2y + 3xy^2 + y^3$. **42.** $8anx^2 - 3nx^3$. **43.** $8ab + 4b + 3c$. **44.** $7ab + 5bc - 3abc$. **45.** $3x^2y + 6xy^2$.

Page 47. — **2.** $16a$. **3.** $3a + b + c$. **4.** $11y - 2z$. **5.** $x^2 + 4z^2$. **6.** $-7az + 9by + 27$. **7.** $4ay^2 + cz^2 + 10$. **8.** $17a^2x^2 + 3b^2y^2$. **9.** $-6a^2bc + 22ab^2c - 7abc^2$. **10.** $2a^2 - 2b^2 + 7d^2$. **11.** $24x^2 + 12y^2$. **12.** $3px - n$. **13.** $-5y^2 + 101z^2$. **14.** $-3ab + 85bc + 2ac$. **15.** $-14x^2 + 75y^2 - 34z^2$.

Page 48. — **16.** $23x^2 + 25pxy - 5y^2$. **17.** $5ax^3 - 4bx^2 + 2cx$. **18.** 7. **19.** $mn(x^2 - x + y)$; $3a(x^3 - x^2 - xy^3 - 2y^4)$. **20.** $x(x - a - b + c) + ab$; $a^2(b - c) - b^2(d - c)$. **21.** $x^2(x - 3y + 5y^2)$; $5(x^2 - 2y) + 6(x + 2y^2)$. **22.** $(a - c)x + (b - d)y$; $a(\sqrt{x} - \sqrt{y} + \sqrt{2z})$.

Page 50. — **10.** $12xy - 8x^2 + 3y$. **11.** $-4xy$. **12.** $2b$. **13.** $4a^2 - 4ab$. **14.** $2a + 2b - 2c$. **15.** $3a^2 - 10ab - 3c$. **16.** $5x^2 + 2xy - 10$. **17.** $6x^2y + 2y^3$. **18.** $-6x^2y - 2y^3$. **19.** $-2a^2 - 3b^2 + 10$. **20.** $2x^3 - 2ax^2 + 8a^2x$. **21.** $2b + 2d - 2e$. **22.** $-2c$. **23.** $-2pv$. **24.** $a^2 + 2ab + 3b^2$. **25.** $8p^2x - 5px^2$. **26.** $9x^3 + 31ax^2 + 18a^3$. **27.** $x^4 - 4x^3y + 12x^2y^2 + 2y^4$.

Page 51. — **28.** $3(x^2 + y^2) + 10(a^2 - b^2)$. **29.** $5\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2} + 5\sqrt{3}$. **30.** $x^3 + x^2 - y^3 - y^2$. **31.** $-3x^2y$. **1.** $9(x + y)$. **2.** $3\sqrt{x - y}$. **3.** $(3a + b)(x^2 - y^2)$. **4.** $15x^2 - (a - x^2)$. **5.** $x - 5\sqrt{xy} + 3y$. **6.** $2a^2 + 3ax + 5x^2$. **7.** $5x^2c - 2xc$. **8.** $(a - b)(x + y + z)$. **9.** $-a(5 - x)$. **10.** $x^2 + 4xy - y^2$. **11.** $3ax^2 - 2bx^2$. **12.** $7x^3y + x^2y^2 + 5xy^3 + 7y^4 + 8$. **13.** $22(a + b) - (a - b)$.

Page 52. — **14.** $-2(x^2 - y^2)$. **15.** $22(a^2 + b^2) + (a^2 - b^2)$. **16.** $22(a + b + c)$. **17.** $2c$. **18.** $6xy^2 + 2y^3$. **1.** $a - b$. **2.** $-a - 2c$. **3.** $4a^2 + ac$.

Page 53. — **4.** $-b^2$. **5.** $4x+5$. **6.** $-x^2-4xy$. **7.** $-x^2+xy$. **8.** $17-x^2$. **9.** a^2+b^2-2ab. **10.** x^4-y^4. **11.** $4ax^3+bx^2+bx+9$. **12.** $4a^2x^2+ax^3$. **13.** $2a^2b^2-2ab$. **14.** a. **15.** $a-b$. **16.** y. **17.** $5m-8n-p$. **18.** $12m-8n-r$. **19.** $-a-7b+2c$. **20.** $6x-11$. **21.** $x+4y$. **22.** $-b$. **23.** $a-b+d$. **24.** $-ab+bc+c$. **25.** x. **26.** $-2+3a-a^2$. **27.** $a+b+7c$. **28.** 18.

Page 56. — **2.** $3a^2c-3abc-3ac^2$. **3.** $5x^2y+5xy^2-5xyz$. **4.** $3a^4b-6a^3b^2+3a^2b^3$. **5.** $12x^4y^2-8x^3y^3+20x^2y^4+4xy^5$. **6.** $-3ac^2x^2-2c^3x^3+5c^2x^4$. **7.** $4a^4bx-24a^5x^2-4a^2bx^4+4a^2x^5$. **8.** $5a^3x^6y^2-5a^4x^5y^2-15a^5x^3y^2+5a^2cx^2y^4$. **9.** $6a^3b^4x^5-42ab^5x^2y^3-6ab^2x^6$. **10.** $6xy^4-2x^2y^3+x^3y^2$. **11.** $-\frac{2}{3}a^2bd-\frac{1}{3}abcd+\frac{2}{3}acd^2$. **12.** $-\frac{3}{8}a^4b^4-\frac{9}{16}a^4b^3+\frac{3}{8}a^3b^3$. **13.** $\frac{2}{3}x^2y^4-\frac{2}{3}x^3y^5+\frac{4}{3}x^4y^6-\frac{2}{3}xy^7$.

Page 58. — **3.** $4a^2+8a^3+2a^4b+3a^4-5a^5b-10a^5b^2-17a^6b^2+15a^7b^3$. **4.** $4x^5-9x^4+31x^3-4x^2+27x+15$. **5.** $x^3+x^2y-6xy^2$. **6.** $2x^3-x^2-16x+15$. **7.** $x^2+7xy+12y^2$. **8.** $5m^4-13m^2n-m^2-6n^2+3n$. **9.** $a^6-a^2b^4$. **10.** $a^3-3a^2b+3ab^2-b^3$. **11.** $ax^2-8a^2x+15a^3$. **12.** x^4-y^4. **13.** $64x^4-9a^2y^2$. **14.** $x^3+x^2y-xy^2-y^3$. **15.** $-m^4n^2+2m^3n^3+3m^2n-m^2n^4-3mn^2$. **16.** $x^4-9x^3+22x^2-24x$. **17.** $x^4-2x^2y^2+y^4$. **18.** $2x^5-10x^4y+9x^3y^2-17xy^4+4y^5$. **19.** $25x^6a^4-30x^5a^5+14x^4a^6-8x^3a^7+3x^2a^8$. **20.** $a^5+a^4-3a^3b+3a^2b^2-a^2b^3-a^4b-ab^3+ab^4$. **21.** $x^5-x^4y-2x^3y^2+2x^2y^3+xy^4-y^5$. **22.** $x^3+3xy+y^3-1$. **23.** $a^6-3a^5-29a^4+101a^3+102a^2-544a+480$. **24.** $z^5-yz^4-2y^2z^3+2y^3z^2+y^4z-y^5$. **25.** $x^6-4x^5+3x^4+2x^3+4x^2-15$.

Page 59. — **1.** x^3+1. **2.** $1-x^4$. **3.** $x^3+3x^2y+3xy^2+y^3$. **4.** $a^3-3a^2b+3ab^2-b^3$. **5.** $x^4-2x^2y^2+y^4$. **6.** $a^4+4a^2b^2+16b^4$. **7.** $15x^4+16x^3y-17x^2y^2+8xy^3-y^4$. **8.** $1-a^2+a^3-2a^4+a^5$. **9.** $25x^6-x^4-2x^3-8x^2$. **10.** x^4-y^4. **11.** a^8-b^8. **12.** $3a^4-26a^3b+37a^2b^2-14ab^3+3a^2-5ab+2b^2$. **13.** $a^6-a^5+a^4b-a^4b^2+a^2b^4-ab^4+b^5-b^6$. **14.** $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$. **15.** $x^7-x^5+x^2-1$. **16.** $35x^4-39x^3+13x^2-26x+4$. **17.** x^4+9x^2+81. **18.** $x^5-41x-120$. **19.** $20x-56x^2+10x^3-32x^4+x^6$. **20.** $3x^8+3x^4y^4+3y^8$. **21.** $a^3-3abc+b^3+c^3$. **22.** $a^4-2a^2b^2+4abc^2+b^4-c^4$.

Page 60. — **23.** x^6-y^6. **24.** x^4-a^4. **25.** $b^4-5b^2c^2+4c^4$. **26.** a^2-b^2; x^2-y^2. **27.** $4a^2-b^2$; $4x^2-9y^2$. **28.** m^2-n^2; $4m^2-n^2$. **29.** $x^3-3x^2-4x+12$. **30.** x^4-2x^2+3x-2. **31.** $x^4+2ax^3-9a^2x^2-2a^3x+8a^4$. **32.** $a^4+2a^3b-2ab^3-b^4$. **33.** $x^4-a^2x^2-2ax^2+2a^3$. **34.** $a^2-2ab+b^2$. **35.** $x^3-3x^2y+3xy^2-y^3$. **36.** $8x^3+12cx^2+6c^2x+c^3$. **37.** $a^4-4a^3b+6a^2b^2-4ab^3+b^4$. **38.** $a^4-4a^3b+6a^2b^2-4ab^3+b^4$. **39.** $x^2-2xy-2xz+y^2+2yz+z^2$. **40.** $27x^3+54x^2y+36xy^2+8y^3$. **41.** $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$. **42.** $3a^2+ab+5ac$. **43.** $a^3-2a^2b-4ab^2-b^3$. **44.** $2ab+ac$. **45.** $3ab$. **46.** $a^2-4ab-3b^2$.

Page 61. — **47.** $6a^2+2$. **48.** $x^4+3x^2y^2+y^4$. **49.** $4x^3y-4xy^3$. **50.** a^3+4a^2+5a+2. **51.** 0.

Page 63. — **5.** $-2b^4c$. **6.** $9a^2y^2$. **7.** $5m^4nx^2$. **8.** $-2ac$. **9.** $8q$. **10.** $40a^3x^2z$. **11.** $7my$. **12.** $-4a^4b$. **13.** $12x$. **14.** $8a^2x^3y$. **15.** $-7nr^2$.

Page 64. — **2.** $-x^2+xy-y^2$. **3.** $-ab+2ab^2-3b^2+b$. **4.** $xy-2x^2y^2-3z$. **5.** $-x^3+2ax^2y-2a^2xy^2+2a^3y^3$. **6.** $2y-4x^3y^2-x^4$. **7.** $-6x^3+2ax^2-10a^2x$. **8.** $-5x+4y-9z$. **9.** $5a^2b^2-4ab^3+3abc^2$. **10.** $-5x+14xy^2z+6x^3y-20yz$.

Page 66. —**5.** $2x^3+4x^2+8x+16$. **6.** $x^3+3x^2y+3xy^2+y^3$. **7.** x^2+xy+y^2. **8.** x^4+y^4. **9.** $2a^2-3ab+4b^2$. **10.** $6a^2b-5ab^2$. **11.** $x+y$. **12.** $x-y$. **13.** $3a+2b$. **14.** $5x-7y$. **15.** $4x+12$. **16.** a^2+ab+b^2. **17.** $1+x+x^2$. **18.** $9x^4+6x^2y+4y^2$. **19.** $25m^4-35m^2+49$. **20.** $m^3-m^2n+mn^2-n^3$. **21.** $a^4+a^3b+a^2b^2+ab^3+b^4$. **22.** $x^4+x^2y^2+y^4$. **23.** $25x^2+10x+1$. **24.** $36x^4-24ax^2+4a^2$. **25.** x^2-5x+4. **26.** $x+a$. **27.** $9x^4-15x^2+25$. **28.** $a^4-a^2b^2+b^4$. **29.** $81x^4+27x^3+9x^2+3x+1$. **30.** $144m^4+84m^2+49$. **31.** $5x^2-3x+9$. **32.** $2x^2-5$. **33.** x^2+5x-1.

Page 67. —**34.** $1-y^2$. **35.** $x^6+x^4y^2+x^2y^4+y^6$. **36.** $x^4-x^2y^2+y^4$. **37.** a^6-a^3+1. **38.** $1-3x+2x^2-x^3$. **39.** $a^2x^4+ab^2x^2y+b^4y^2$. **40.** $a+b$. **41.** $4x^2-12x+9$. **42.** $1+2y+3y^2+2y^3+y^4$. **43.** $3x^2-xy-2y^2$. **44.** $4+3x+2x^2$. **45.** $x^4-5x^3-11x^2+99x+10$.

Page 68. —**1.** $\frac{3}{4}ab^2+\frac{1}{6}b^2c+\frac{1}{15}$. **2.** $2a^2-\frac{5}{4}ab+\frac{4}{3}b^2$. **3.** $\frac{1}{6}x-\frac{1}{4}^3y+\frac{5}{4}xy$. **4.** $\frac{1}{6}a^2+\frac{1}{12}ab+b^2$. **5.** $\frac{1}{4}x^3+\frac{1}{72}x-\frac{1}{12}$. **6.** $\frac{1}{4}a^3-\frac{5}{6}a^2+\frac{1}{12}a+\frac{1}{2}$. **7.** $\frac{1}{2}a+\frac{1}{3}b$. **8.** $\frac{1}{9}x^2-\frac{1}{6}x+\frac{1}{16}$. **9.** $\frac{1}{6}x^3-\frac{5}{12}x^2-\frac{5}{24}x+\frac{3}{16}$. **10.** $\frac{2}{5}a^3-\frac{5}{18}ab^2-\frac{1}{3}b^3$. **11.** $x^3+\frac{1}{6}x^2+\frac{1}{6}x-\frac{1}{9}$. **12.** $\frac{1}{2}x-\frac{1}{3}$. **13.** $\frac{1}{2}a+\frac{1}{3}b$. **14.** $\frac{1}{2}x^2-\frac{2}{3}x-\frac{3}{4}$. **15.** $\frac{1}{2}x+\frac{1}{3}y$.

Page 69. —**5.** $a^4-2a^2b^2+b^4$. **6.** $2-5x+10x^2-5x^3+5x^4+2x^5$. **7.** $6x^5-26x^4+49x^3-56x^2+33x-10$. **8.** $x^8+x^4y^4+y^8$. **9.** $a^6-6a^5+13a^4-14a^3+10a^2-4a+1$. **10.** $a^5-41a-120$. **11.** $8a^6-6a^5-5a^4-16a^3-12a^2-6a-3$. **12.** $x^5+\frac{1}{4}x^4+1\frac{3}{2}x^3+\frac{1}{4}\frac{3}{8}x^2-\frac{1}{3}x-\frac{1}{12}$.

Page 70.—**13.** $amx^6-anx^5+(bm-ar)x^4+(cm-bn)x^3+(dm-cn-br)x^2-(dn+cr)x-dr$. **14.** $x^6-6x^5-3x^4+40x^3+24x^2-24x+4$. **15.** $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. **19.** x^2-3x-1. **20.** $2x^2+3x+2$.

Page 73. —**4.** x^2-1. **5.** x^2-3x-1. **6.** x^2+x+2. **7.** $1+x+2x^2$. **8.** x^3-x^2-3. **9.** x^2+5x-1. **10.** $x^3+3x^2y+3xy^2+y^3$. **11.** $x^2-\frac{1}{2}x-\frac{1}{8}$.

Page 77. —**3.** $x=10$. **4.** $x=5$. **5.** $x=5$. **6.** $x=3$. **7.** $x=6$. **8.** $x=4$. **9.** $x=3$. **10.** $x=5\frac{1}{2}$. **11.** $x=7$. **12.** $x=6$. **13.** $x=4$. **14.** $x=9$.

Page 78. —**15.** $x=1$. **16.** $x=10$. **17.** $x=7$. **18.** $x=9$. **19.** $x=4$. **20.** $x=\frac{1}{4}$. **21.** $x=6$. **22.** $x=5$. **23.** $x=1$. **24.** $x=5$. **25.** $x=3$. **26.** $x=2$. **27.** $x=9$. **28.** $x=14$. **29.** $x=15$. **30.** $x=2$. **31.** $x=6$. **32.** $x=b$. **33.** $x=2$. **34.** $x=1$. **35.** $x=-1$. **36.** $x=4a$. **37.** $x=3$. **38.** $x=b$. **39.** $x=\dfrac{3}{a^2}$. **40.** $x=2-a^2-b$. **41.** $x=a+b$. **42.** $x=a+b$. **43.** $x=2b$. **44.** $x=ab$.

Page 81. —**15.** 38 and 55. **16.** 6. **17.** 22 and 32. **18.** Man's age, 25 yrs.; brother's, 30 yrs. **19.** 15 yrs. **20.** Son's age, 9 yrs.; father's, 36 yrs. **21.** B, \$252; A, \$324; C, \$424.

Page 82.—**22.** Horse, \$120; buggy, \$80. **23.** Less, 27; greater, 45. **24.** Boys, 78; girls, 65. **25.** 15. **26.** First, 4; second, 12; third, 24. **27.** Unsuccessful candidate, 1030; successful, 1530. **28.** A's share, \$1243; B's, \$904. **29.** 112 sheep. **30.** A, \$35; B, \$40. **31.** 50 pieces. **32.** Horses, 4; cows, 12; sheep, 80. **33.** Half dollars, 60; dimes, 40.

Page 83. —**34.** 60 children. **35.** A, \$3500; B, \$4000; C, \$7500. **36.** Eldest, 24 yrs.; second, 21 yrs.; third, 18 yrs.; fourth, 15 yrs.; youngest, 12 yrs. **37.** Less, 7; greater, 11. **38.** C's share, \$440; B's, \$1320; A's, \$2640. **39.** Women, 18; men, 22; children, 50. **40.** 5 vessels. **41.** First, \$1300; second, \$1500; third, \$1300; fourth, \$900. **42.** 103 gals. **43.** Sister's age, 6 yrs.; brother's, 12 yrs.

Page 84. — **44.** Artillery, 250 men; cavalry, 450 men; infantry, 3800 men. **45.** 19 hours. **46.** 50 and 20. **47.** $2400. **48.** Left pocket, 15; right, 20. **49.** Youngest, $425; third, $475; second, $525; oldest, $575. **50.** Quarters, 5; dimes, 15; five-cent pieces, 75. **51.** 45 miles. **52.** 5 miles.

Page 85. — **4.** x^2+6x+9. **5.** $25+10x+x^2$. **6.** $9a^2+6ab+b^2$. **7.** $16x^2+8xy+y^2$. **8.** $4a^2x^2+12ax+9$. **9.** $9a^2+30ab+25b^2$.

Page 86. — **13.** $25-10x+x^2$. **14.** $a^4-2a^2b^2+b^4$. **15.** $9a^2-6ab+b^2$. **16.** $x^2-10xy+25y^2$. **17.** $a^2x^2-8ax+16$. **18.** $16a^2-24ab+9b^2$.

Page 87. — **19.** $a^2+6ax+9x^2$. **20.** $x^2-6xy+9y^2$. **21.** $4m^2+4m+1$. **22.** $1-4m+4m^2$. **23.** $a^4-4a^2b+4b^2$. **24.** $4a^2-12ab+9b^2$. **25.** $9a^4-12a^2b^2+4b^4$. **26.** $a^2x^2-4axy+4y^2$. **27.** $a^2b^2+4abc^2+4c^4$. **28.** $a^6b^2-4a^3bcx+4c^2x^2$. **29.** $x^8y^2+50x^4y+625$. **30.** $16a^2b^2-16abxy+4x^2y^2$. **33.** $9x^2-9$. **34.** $4a^2-9b^2$. **35.** $a^2x^2-b^2$. **36.** $1-9x^4y^2$. **37.** $16a^4x^2-9y^2$. **38.** $25a^4b^4-1$.

Page 88. — **41.** $x^2+y^2+z^2+2xy-2xz-2yz$. **42.** $x^2+y^2+z^2-2xy+2xz-2yz$. **43.** $x^2+y^2+1+2xy-2x-2y$. **44.** $x^2+y^2+z^2-2xy-2xz+2yz$. **45.** $4a^2+b^2+c^2-4ab+4ac-2bc$. **46.** $a^2+9b^2+c^2+6ab-2ac-6bc$. **47.** $x^2+y^2+z^2+v^2+2xy+2xz+2vx+2yz+2vy+2vz$. **48.** $x^2+y^2+z^2+v^2-2xy-2xz-2vx+2yz+2vy+2vz$. **49.** $a^2+4b^2+c^2+4d^2-4ab-2ac+4ad+4bc-8bd-4cd$. **50.** $a^2+b^2+c^2+d^2+2ab-2ac-2ad-2bc-2bd+2cd$. **51.** $a^4+b^4+c^4+d^4+2a^2b^2-2a^2c^2-2a^2d^2-2b^2c^2-2b^2d^2+2c^2d^2$. **52.** $a^4+b^4+4c^2+d^2+2a^2b^2-4a^2c+2a^2d-4b^2c+2b^2d-4cd$.

Page 89. — **53.** $x^2+7x+12$. **54.** $x^2-10x+24$. **55.** $x^2+5x-24$. **56.** $x^2-2x-63$. **57.** $x^2-3x-18$. **58.** $x^2-8x-65$. **59.** $x^2-16x+60$. **60.** $x^2-3x-108$. **61.** $x^2-24x+135$. **62.** $x^2-(a+b)x+ab$. **63.** $a^2+5a-66$. **64.** $81-9(x+y)+xy$. **65.** $x^2-(2a+3)x+6a$. **66.** $x^2+2cx-15c^2$. **67.** $x^2-4(b+1)x+16b$. **68.** $x^2+abx-2a^2b^2$.

Page 90. — **69.** $3x^2-26x+35$. **70.** $6x^2-11x-35$. **71.** $12m^2-31m+20$. **72.** $3x^2-10x+3$. **73.** $3x^2-5x-2$. **74.** $5x^2-47x+84$. **75.** $6x^2-xy-2y^2$. **76.** $6-13ab+6a^2b^2$. **77.** $2a^2-ab-3b^2$. **78.** $5x^2+6xy-8y^2$. **79.** $2m^2+m^2n-3m^2n^2$. **80.** $3a^2+ab-2b^2$. **81.** $3x^2-5bx-2b^2$. **82.** $10y^2-12ay+2a^2$. **83.** $4x^2+4xy-3y^2$. **84.** $15a^2+4ab-35b^2$. **1.** $x-y$. **2.** a^2+b. **3.** $x-3$. **4.** $3a+b$. **5.** $2ax+3$. **6.** $5x-y$. **7.** $3a-2b$. **8.** $2x^2-3y^2$. **9.** $x+8$. **10.** $a-10$. **11.** $x-2a$. **12.** $x-3c$.

Page 91. — **13.** $x+y$. **14.** x^2-y^2. **15.** x^2+4. **16.** $3x-3$. **17.** $2a+3b$. **18.** $3ax+b$. **19.** $10+3ab$. **20.** $1+3x^2y$. **21.** $5ax^2-1$. **22.** $4x^2y+3$.

Page 92. — **23.** x^2-xy+y^2. **24.** x^2+xy+y^2. **25.** $4a^2-2ab+b^2$. **26.** $4a^2+2ab+b^2$. **27.** $x^2+3bx+9b^2$. **28.** $1+2y+4y^2$. **29.** $4y^2+2y+1$. **30.** $x^2y^2+xyz+z^2$. **31.** $x^2y^2-xyz+z^2$. **32.** $9-3xy+x^2y^2$. **33.** $x^4+x^2y^2+y^4$. **34.** $x^4-x^2y^2+y^4$. **35.** $1+4xy+16x^2y^2$. **36.** $9x^2y^2+6xy+4$.

Page 93. — **37.** $x^3-x^2y+xy^2-y^3$. **38.** $x^3+x^2y+xy^2+y^3$. **39.** $x^4-x^3y+x^2y^2-xy^3+y^4$. **40.** $x^4+x^3y+x^2y^2+xy^3+y^4$. **41.** $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$. **42.** $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$. **43.** x^3+x^2+x+1. **44.** $1-x+x^2-x^3$. **45.** $1-x+x^2-x^3+x^4$. **46.** $8+4x$

$+2x^2+x^3$. **47.** $x^5-2x^4+4x^3-8x^2+16x-32$. **48.** $x^3+3x^2+9x+27$. **49.** $4x^2-2x+1$. **50.** $x^2-3xy+9y^2$. **1.** $x^2+2xy+y^2$. **2.** $4x^2-4xy+y^2$. **3.** m^2+2m+1. **4.** $1-2m+m^2$. **5.** $4-12a+9a^2$. **6.** $4x^2-12xy+9y^2$. **7.** $9x^4-12x^2y^2+4y^4$. **8.** $a^4+4a^2b+4b^2$. **9.** $25-10bc+b^2c^2$. **10.** $4a^2x^2+4axy^2+y^4$. **11.** $a^2b^2-2abc+c^2$. **12.** $49-28ax+4a^2x^2$. **13.** $c^2+d^2+e^2+2cd+2ce+2de$. **14.** $a^2+b^2+c^2-2ab+2ac-2bc$. **15.** $x^2+y^2+z^2+2xy-2xz-2yz$. **16.** $1+c^2+d^2+2c-2d-2cd$. **17.** $9a^2+b^2+1-6ab+6a-2b$. **18.** $a^2+4b^2+1-4ab+2a-4b$. **19.** $x^2+4y^2+9-4xy+6x-12y$. **20.** $a^2+4b^2+9c^2+4ab+6ac+12bc$. **21.** $25a^2+b^2+c^2-10ab-10ac+2bc$. **22.** $m^2+n^2+r^2+s^2+2mn+2mr+2ms+2nr+2ns+2rs$. **23.** $m^2+n^2+r^2+s^2-2mn+2mr-2ms-2nr+2ns-2rs$. **24.** $a^2+b^2+c^2+1+2ab-2ac-2a-2bc-2b+2c$.

Page 94. — **25.** x^2-y^2. **26.** m^4-n^2. **27.** $4y^4-1$. **28.** $1-9a^2x^2$. **29.** $25-30x+9x^2$. **30.** $49a^2+70x+25$. **31.** $16a^4-4b^2$. **32.** x^4-y^4. **33.** x^6-y^6. **34.** $a^2+3a-28$. **35.** $x^2-7ax+12a^2$. **36.** $x^2+(b-c)x-bc$. **37.** $x^2-(m-n)x-mn$. **38.** $m^2+7m-60$. **39.** $m^2+(r+s)m+rs$. **40.** x^3+1. **41.** x^3-1. **42.** x^4-1. **43.** a^3+x^3. **44.** x^3-a^3. **45.** x^3+8. **46.** x^3-27. **47.** x^4-y^4. **48.** x^4-y^4. **49.** x^2-xy+y^2. **50.** a^2+ax+x^2. **51.** $x^3+x^2y+xy^2+y^3$. **52.** $a^3-a^2x+ax^2-x^3$. **53.** $a^2+4a+16$. **54.** $a^3-3a^2+9a-27$. **55.** x^2-3. **56.** $1-a^2$. **57.** a^3+a^2+a+1. **58.** $x^4+x^3y+x^2y^2+xy^3+y^4$. **59.** $4x^2-2xy+y^2$. **60.** $4a^2+6ab+9b^2$. **61.** $3ax-4b^2$. **62.** $2m^2n-5$. **63.** $a+3$. **64.** $x-4$. **65.** a^2-b^2. **66.** x^2-y^4.

Page 96. — **11.** $5a^3x^2$, $5a^3x^2$. **12.** $9x^2y^4$, $9x^2y^4$. **13.** $11m^2n^4r^3$, $11m^2n^4r^3$. **14.** $7mn^2s^5$, $7mn^2s^5$. **15.** $8ab^3c^2$, $8ab^3c^2$. **16.** $12xyz^2$, $12xyz^2$.

Page 97. — **3.** $5ab(2a+b)$. **4.** $7xyz(2x-9y)$. **5.** $ax(x^2-axy+y^2)$. **6.** $a^2b(a^2-ab+b^2)$. **7.** $a^2(ax^2+y-1)$. **8.** $3a^2b(3x-5y-1)$. **9.** $5x^3(3y-2y^2+z)$. **10.** $4x^4y(1-3y-4y^2+2y^3)$. **11.** $3x^2y^2(x-2x^2+3y^2-4xy)$. **12.** $ay^2(1-3az+5a^2z^2-a^3y^2z^3)$.

Page 98. — **2.** $(2x+y)(2x+y)$ **3.** $(b+x)(a+y)$. **4.** $(a-y)(a+x)$. **5.** $(y-b)(a+x)$. **6.** $(x-2m)(y-n)$. **7.** $(a+b)(x^2-y^2)$. **8.** $(y^2+1)(y+1)$. **9.** $(a-b)(a^2+3)$. **10.** $a(y-b)(x^2-y^2)$. **11.** $(3a+2)(2a^2-3)$. **12.** $(2a^2-b)(3x^2-2y^2)$. **2.** $(2x+y)(2x+y)$. **3.** $(a+3b)(a+3b)$. **4.** $(a^2-b^2)(a^2-b^2)$. **5.** $(3x-y)(3x-y)$. **6.** $(2x^2-5y^2)(2x^2-5y^2)$. **7.** $(x-1)(x-1)$.

Page 99. — **8.** $(2y-1)(2y-1)$. **9.** $(3x-4y)(3x-4y)$. **10.** $(5x-y^2)(5x-y^2)$. **11.** $(12x^2-5y)(12x^2-5y)$. **12.** $(1-5a^2y^2)(1-5a^2y^2)$. **13.** $(5+3ab)(5+3ab)$. **14.** $(2-10ab)(2-10ab)$. **15.** $(xy^2-3)(xy^2-3)$. **16.** $(4+5ab^2c^3)(4+5ab^2c^3)$. **17.** $(11a+100b^4)(11a+100b^4)$. **18.** $(m^2n-20s^2)(m^2n-20s^2)$. **19.** $(a+b-2)(a+b-2)$. **20.** $(x-y-3)(x-y-3)$. **21.** $(m^2-6n)(m^2-6n)$. **23.** $(a+b)(a-b)(x+y)(x+y)$. **24.** $(x+y+z)(x+y+z)$. **25.** $(x-y+z)(x-y+z)$. **26.** $(a^2-2b+2)(a^2-2b+2)$.

Page 100. — **4.** $(3a^2b+4c)(3a^2b-4c)$. **5.** $(xy+2yz)(xy-2yz)$. **6.** $(x^2+y^2)(x+y)(x-y)$. **7.** $(3y+1)(3y-1)$. **8.** $(2ax+6a)(2ax-6a)$. **9.** $(4a^2x^2+7z)(4a^2x^2-7z)$. **10.** $(1+9z^2)(1+3z)(1-3z)$. **11.** $(y^2+4z^2)(y+2z)(y-2z)$. **12.** $(11+a^2x^4)(11-a^2x^4)$. **13.** $(x^2+7y^2z)(x^2-7y^2z)$. **14.** $(12+5ay)(12-5ay)$. **15.** $(x^2+a-b)(x^2-a+b)$. **16.** $(a+x-y)(a-x+y)$. **17.** $(2x^2+a+b)(2x^2-a-b)$.

Page 101. — **18.** $4xy$. **19.** $(a+b+c+d)(a+b-c-d)$. **20.** $-4mn$. **21.** $(a-b+c-d)(a-b-c+d)$. **22.** $8ab(a^2+b^2)$. **23.** $(5a-1)(7-a)$. **24.** $(5a-b)(a+5b)$. **25.** $(6x-y)(4x+5y)$. **29.** $(2a-b+xy)(2a-b-xy)$. **30.** $(2x^2+3x-1)(2x^2-3x+1)$. **31.** $(6ab-c)(-4ab-c)$. **32.** $(a-b+2xy)(a-b-2xy)$. **33.** $(1-x+4ab^2)(1-x-4ab^2)$. **34.** $(a-2b+2x+3y)(a-2b-2x-3y)$. **35.** $(a+1+b-c)(a+1-b+c)$. **36.** $(x-y+m+n)(x-y-m-n)$. **37.** $(x+y+s-z)(x+y-s+z)$. **38.** $(a-c+b+d)(a-c-b-d)$.

Page 102. — **41.** $(x^2+y^2+xy)(x^2+y^2-xy)$. **42.** $(x^2+3y^2+2xy)(x^2+3y^2-2xy)$. **43.** $(m^2-2n^2+2mn)(m^2-2n^2-2mn)$. **44.** $(a^4+b^2+a^2b)(a^4+b^2-a^2b)$. **45.** $(x^4+3y^4+2x^2y^2)(x^4+3y^4-2x^2y^2)$. **46.** $(2x^2-3y^2+2xy)(2x^2-3y^2-2xy)$.

Page 103. — **48.** $(x^2+2+2x)(x^2+2-2x)$. **49.** $(8+y^2+4y)(8+y^2-4y)$. **50.** $(x+4)x$. **51.** $(x^2+2y^2)x^2$. **52.** $(a^2+2b^2+ab)(a^2+2b^2-ab)$. **53.** $(a^2-b^2+2ab)(a^2-b^2-2ab)$.

Page 104. — **3.** $(x+5)(x+2)$. **4.** $(x-11)(x-2)$. **5.** $(x-9)(x-5)$. **6.** $(x+4)(x+5)$. **7.** $(x+7)(x+8)$. **8.** $(x-7)(x-4)$. **11.** $(x+8)(x-3)$. **12.** $(x-12)(x+5)$. **13.** $(x-9)(x+5)$. **14.** $(x+9)(x-7)$. **15.** $(x+14)(x-4)$. **16.** $(x-10)(x+7)$. **17.** $(x-9)(x+8)$. **18.** $(x+7)(x-6)$.

Page 105. — **19.** $(x+4)(x+4)$. **20.** $(x+10)(x+4)$. **21.** $(x+9)(x+7)$. **22.** $(x+12)(x+6)$. **23.** $(y+12)(y+8)$. **24.** $(x-9a)(x-6a)$. **25.** $(x-5a)(x-3a)$. **26.** $(x-7)(x-8)$. **27.** $(x-13)(x-4)$. **28.** $(y-20)(y-7)$. **29.** $(x+7)(x-4)$. **30.** $(x+9)(x-2)$. **31.** $(x+11)(x-6)$. **32.** $(x+12)(x-11)$. **33.** $(x+14)(x-13)$. **34.** $(xy-11)(xy+2)$. **35.** $(xy+8)(xy-13)$. **36.** $(x-12a)(x+5a)$. **37.** $(x-11)(x+9)$. **38.** $(z-17)(z+16)$. **39.** $(x+25)(x+7)$. **40.** $(x+15)(x-5)$. **41.** $(x-20)(x-20)$. **42.** $(x+13)(x-12)$. **43.** $(y+11)(y+6)$. **44.** $(x-19)(x-4)$. **45.** $(x+3)(x-18)$. **46.** $(y+14)(y-3)$. **47.** $(y-17)(y+10)$. **48.** $(y+18)(y+6)$. **49.** $(x^2+11)(x^2+6)$. **50.** $(x-2)(x^2+2x+4)(x^3-7)$. **51.** $(x^4+11)(x^2+3)(x^2-3)$. **52.** $(y+2)(y^2-2y+4)(y-1)(y^2+y+1)$. **53.** $(y^5-16)(y^5-3)$. **54.** $(x+b)(x+c)$. **55.** $(x+a)(x-c)$. **56.** $(y+a)(y-a)(y+b)(y-b)$.

Page 106. — **58.** $(x-16)(x+15)$. **59.** $(x+16)(x+15)$. **60.** $(y+23)(y-13)$. **61.** $(y-30)(y+16)$. **62.** $(x-27)(x-20)$. **63.** $(x-21)(x+20)$. **64.** $(x+36)(x-35)$. **65.** $(x+30)(x+8)$. **66.** $(x+40)(x-33)$. **67.** $(x+24)(x-23)$.

Page 107. — **4.** $(3x-7)(x-5)$. **5.** $(3x+5)(2x-7)$. **6.** $(4m-5)(3m-4)$. **7.** $(3x-1)(x-3)$. **8.** $(3x+1)(x-2)$. **9.** $(5x-12)(x-7)$. **10.** $(3x+b)(x-2b)$. **11.** $(3x-2y)(2x+y)$. **12.** $(5x-4y)(x+2y)$. **13.** $(2m+3mn)(m-mn)$. **14.** $(3a-2b)(a+b)$. **15.** $2(y-a)(5y-a)$. **16.** $(2x-y)(2x+3y)$. **17.** $(5x+7y)(3x-5y)$.

Page 108. — **20.** $(x+3)(3x+5)$. **21.** $(x+2)(7x+6)$. **22.** $(5x-12)(x-1)$. **23.** $(2x+3)(3x-2)$. **24.** $(x-2)(3x+5)$. **25.** $(3x-8)(2x+5)$. **26.** $(2x-3)(3x-2)$. **27.** $(x+3)(3x-4)$. **28.** $(x-4)(3x-5)$. **29.** $(x+3)(10x-7)$.

Page 109. — **2.** $(x+5)(x+4)$.

Page 110. — **5.** $(x+3)(3x-4)$. **6.** $(x+3)(x+4)$. **7.** $(2x-7)(x+4)$. **8.** $(3x-7)(x-2)$. **9.** $(2x+m)(x-3m)$. **10.** $(3x+n)(x-2n)$.

Page 112. —**5.** $(x+2)(x^2-2x+4)$. **6.** $(x-1)(x^2+x+1)$. **7.** $(1+x)(1-x+x^2)$. **8.** $(x-3)(x^2+3x+9)$. **9.** $(5+x^2)(25-5x^2+x^4)$. **10.** $(4+x^2)(2+x)(2-x)$. **11.** $(x^2+16)(x+4)(x-4)$. **12.** $(a^2-3)(a^4+3a^2+9)$. **13.** $(x^2-y)(x^4+x^2y+y^2)$. **14.** $(m+n)(m^2-mn+n^2)(m-n)(m^2+mn+n^2)$. **15.** $(m^2+n^2)(m^4-m^2n^2+n^4)$. **16.** $(x-1)(x^4+x^3+x^2+x+1)$. **17.** $(1+a+b)(1-a-b+a^2+2ab+b^2)$. **18.** $(a-b+1)(a^2-2ab+b^2-a+b+1)$. **19.** $8mn(m^2+n^2)$. **20.** $2n(3m^2+n^2)$. **21.** $3(a+x)(a-x)(5a^2+8ax+5x^2)$. **22.** $(a-2x)(7a^2+8ax+4x^2)$. **23.** $(x+y+\sqrt{2xy})(x+y-\sqrt{2xy})$. **24.** $(a+b^2+\sqrt{2ab^2})(a+b^2-\sqrt{2ab^2})$. **25.** $(m^2+n^4+\sqrt{2m^2n^4})(m^2+n^4-\sqrt{2m^2n^4})$. **26.** $(9+x^2+\sqrt{18x^2})(9+x^2-\sqrt{18x^2})$. **27.** $(a^4+b^4+\sqrt{2a^4b^4})(a^4+b^4-\sqrt{2a^4b^4})$. **28.** $(3x^2+2y+\sqrt{12x^2y})(3x^2+2y-\sqrt{12x^2y})$. **29.** $(2x+y^2+\sqrt{4xy^2})(2x+y^2-\sqrt{4xy^2})$. **30.** $(x^2+16+\sqrt{32x^2})(x^2+16-\sqrt{32x^2})$.

Page 115. —**7.** $(x-1)(x+2)(x+3)$. **8.** $(x+1)(x+2)(x-3)$. **9.** $(x+2)(x+3)(x-5)$. **10.** $(x-1)(x-6)(x+7)$. **11.** $(x-2)(x-3)(x-5)$. **12.** $(x-1)(x+3)(x+7)$. **13.** $(x-2)(x-2)(x+3)$. **14.** $(x+1)(x-3)(3x-4)$. **15.** $(x-1)(x-2)(2x-3)$. **16.** $(x+1)(6x^2-6x+13)$. **17.** $(x+3)(2x^2-6x+13)$. **18.** $(x-1)(x-2)(x+3)(x-4)$. **19.** $(x+2)(x-2)(x^2+x+4)$. **20.** $(x+2)(x-2)(x^2-3x+1)$.

Page 116. —**1.** $(3x-1)(3x-1)$. **2.** $xy(x+y)(x+y)$. **3.** $2x(x-2)(x-2)$. **4.** $5a^2(x-3y^2)(x-3y^2)$. **5.** $x(x-5)(x-5)$. **6.** $12ax(ax-4b)(ax-4b)$.

Page 117. —**7.** $3a(m-n)(m-n)$. **8.** $5a(a-b)(a-b)$. **9.** $(x^2+1)(x+1)$. **10.** $3ab(x^2-y^2+8)$. **11.** $7x^2(x-3y)(x-3y)$. **12.** $a(a+1)(a-1)$. **13.** $(2xy+z)(2xy-z)$. **14.** $7x^2(x-3y+2xy^2)(x-3y-2xy^2)$. **15.** $5aby^2(x+ay)(x+ay)$. **16.** $9(a^2+2)(a^2-2)$. **17.** $(x^2+1)(x+1)(x-1)$. **18.** $(a^2+b^2)(a+b)(a-b)$. **19.** $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$. **20.** $(5+3x^3)(5-3x^3)$. **21.** $(1+m^2)(1+m)(1-m)$. **22.** $(a-b+c-d)(a-b-c+d)$. **23.** $4mn$. **24.** $x(2x+3)(2x-3)$. **25.** $5s(x^2+3y^2)(x^2-3y^2)$. **26.** $5(x-y)(x+y)$. **27.** $(n^4-8b^4+9a^2b^2)(n^4-8b^4-9a^2b^2)$. **28.** $(5x-8)(x-2)$. **29.** $3x^2y^2(x+3y)(x-3y)$. **30.** $3ay(x-3y)(x-3y)$. **31.** $5x(a-b+3x)(a-b-3x)$. **32.** $(1-3x+2xy^2)(1-3x-2xy^2)$. **33.** $(x+d)(x+c)$. **34.** $(x+d)(x-c)$. **35.** $(x-n)(x-m)$. **36.** $(a-d)(b+c)$. **37.** $(2c-d)(a-2b)$. **38.** $(3x-y)(s-3z)$. **39.** $(x+3)(x-y)$. **40.** $(1+a-bc^2)(1-a+bc^2)$. **41.** $(a^2+c^2+ac)(a^2+c^2-ac)$. **42.** $(x^2+1+x)(x^2+1-x)$. **43.** $(m^2+n^2+mn)(m^2+n^2-mn)$. **44.** $4(x^2+2+2x)(x^2+2-2x)$. **45.** $9(a^2+2+2a)(a^2+2-2a)$. **46.** $(m^2+8n^2+4mn)(m^2+8n^2-4mn)$. **47.** $(x^2+2y^2+2xy)(x^2+2y^2-2xy)$. **48.** $(2m^2+n^2+2mn)(2m^2+n^2-2mn)$. **49.** $(2a+3b)(2a-3b)$. **50.** $(x^2+2y^2+xy)(x^2+2y^2-xy)$. **51.** $(x^2+y^2+xy)(x^2+y^2-xy)$. **52.** $(a^2+2b^2)(a^2-2b^2)$. **53.** $(2+a^2)(2-a^2)$. **54.** $(1-x)(1+x+x^2)$. **55.** $(2-y)(4+2y+y^2)$. **56.** $(x-y)(x^2+xy+y^2)$. **57.** $(a+b)(a^2-ab+b^2)$. **58.** $(1+a)(1-a+a^2)$. **59.** $(x-1)(x^2+x+1)$. **60.** $(m+n)(m-n)(m^2-mn+n^2)(m^2+mn+n^2)$. **61.** $(2a^2-3b^2)(4a^4+6a^2b^2+9b^4)$. **62.** $(3+x)(9-3x+x^2)$. **63.** $(a^2+2)(a^4-2a^2+4)$. **64.** $(a^2+b^2)(a^4-a^2b^2+b^4)$. **65.** $(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)$. **66.** $(a-x)(a^4+a^3x+a^2x^2+ax^3+x^4)$.

Page 118. — **67.** $(x+2)(x-2)(x^2-2x+4)(x^2+2x+4)$. **68.** $(1+m)(1-m)(1-m+m^2)(1+m+m^2)$. **69.** $(3-x)(9+3x+x^2)$. **70.** $(1+x+y)(1-x-y+x^2+2xy+y^2)$. **71.** $(x-y+1)(x^2-2xy+y^2-x+y+1)$. **72.** $(x^2-11)(x+1)(x-1)$. **73.** $(x^2-7)(x+1)(x-1)$. **74.** $(a+3)(a-3)(a+1)(a-1)$. **75.** $(x+12)(x-10)$. **76.** $(x+24)(x-3)$. **77.** $(x+16)(x-15)$. **78.** $(x-18)(x+8)$. **79.** $(x+17)(x+13)$. **80.** $(7x+1)(3x-1)$. **81.** $(4x-7)(9x+8)$. **82.** $(x-21)(x-20)$. **83.** $(x-15)(x-12)$. **84.** $x(x-5)(x+4)$. **85.** $(4a-2)(a-3)$. **86.** $(4a^2-2)(a^2-2)$. **87.** $(3x+13)(x-5)$. **88.** $3(x^2-2x-5)$. **89.** $(x-7)(x+1)$. **90.** $(x-5)(x-5)$. **91.** $(x-7)(x+2)$. **92.** $(x^2-12)(x^2+5)$. **93.** $(x^2-10)(x^2+4)$. **94.** $(3x+2y)(3x+2y)(3x-2y)(3x-2y)$. **95.** $(x+n)(x-n)(x^2+m^2)$. **96.** $(x+a+1)(x+a+1)$ **97.** $(a+y)(a-y)(a^2-ay+y^2)$. **98.** $(x^2-3m)(x^2+2)$. **99.** $(x+y-z)(x+y-z)$. **100.** $(a-b+c+d)(a-b-c-d)$. **101.** $(m-n+a-b)(m-n-a+b)$. **102.** $(a-2+m-n)(a-2-m+n)$. **103.** $(3x-y+2s+2z)(3x-y-2s-2z)$. **104.** $(x-y+1)(x-y+1)$. **105.** $(ax-b)(x+y)(x+y)$. **106.** $(x^2+y^2)(x+y)(x-y)(a-b)(a-b)$. **107.** $(m-n)(x-y)(x-y)$. **108.** $(x-1)(x-3)(x-5)$. **109.** $(x-1)(x+1)(x+3)(x-5)$. **110.** $(x+5)(x+2)(x+1)(x-1)$.

Page 121. — **6.** 25. **7.** $15ab^2c$. **8.** $8x^2y^2$. **9.** $6a^3x^2y^3$. **10.** $x-y$. **11.** $x-1$. **12.** x^3+ax^2. **13.** $15(x-1)$. **14.** $4(a^2-b^2)$. **15.** a^2-ab+b^2. **16.** $x-1$. **17.** x^2+x+1. **18.** x^2-4x+4. **19.** $y+2z$. **20.** x^2+y. **21.** $x+4$. **22.** $x-9$. **23.** $x+7$. **24.** $x-12$. **25.** $x+1$. **26.** x^2+12.

Page 122. — **27.** $x-1$. **28.** $x-5$. **29.** $2x+3$. **30.** $x-1$.

Page 126. — **6.** $x-5$. **7.** $2x+3$. **8.** $x+5$. **9.** $x-4$. **10.** $x-3$. **11.** $x-6$. **12.** $x-4$. **13.** $x^2-9x+21$. **14.** x^2+x+1. **15.** $x+3$. **16.** x^2-x-5. **17.** x^2-4x+3. **18.** $2x-5$. **19.** $2x^2-x+3$. **20.** x^2-y^2.

Page 128. — **4.** $15a^3x^4y$. **5.** $126x^2y^3z^3$. **6.** $180a^5b^3c^3d^5x^2$. **7.** $(x+y)(x-y)(x^2+xy+y^2)$. **8.** $35xy^2(a+b)(a+b)(a^2-ab+b^2)$. **9.** $a^2(a-b)(a-4b)(a^2+ab+b^2)$. **10.** $(x-2)(x-5)(x+5)$. **11.** $(x+10)(x-11)(x-9)(2x-5)$. **12.** $(x+5)(x+3)(x-1)$. **13.** $(x-12)(x+7)(x+5)$. **14.** $(7x-1)(x-5)(x+5)$. **15.** $ab(a+b)(a-b)$. **16.** $(x^2+y^2)(x+y)(x+y)(x-y)(x-y)$. **17.** $(x-1)(x+1)(x^2-x+1)(x^2+x+1)$.

Page 129. — **3.** H. C. F., $2x-3$; L. C. M., $(3x-4)(12x^2-4x-21)$. **4.** H. C. F., $x-5$; L. C. M., $x(x-5)(x^2+5x+2)(7x^2+x+1)$. **5.** H. C. F., $x+1$; L. C. M., $(x+1)(3x^2-11x-1)(5x^2-17x-7)$. **6.** H. C. F., $x-1$; L. C. M., $(x-1)(3x^2+3x-2)(21x^2+x-10)$. **7.** H. C. F., $4x-3$; L. C. M., $(4x-3)(x^2+4)(3x^2-2x+4)$. **8.** H. C. F., $x+1$; L. C. M., $x^2(x+1)(x^2-x-1)(x^2+x+1)$. **9.** H. C. F., x^2-5x+6; L. C. M., $(x-2)(x-3)(x-1)(x-4)$. **10.** H. C. F., $2x-3$; L. C. M., $(2x-3)(3x-2)(x+4)(3x+4)$.

Page 132. — **1.** $\frac{3x}{36a}$. **2.** $\frac{m^2}{3m}$. **3.** $\frac{2x^2}{x+xy}$. **4.** $\frac{1}{3m^2}$. **5.** $x+y$. **6.** $\frac{a^3+a^2b+ab^2+b^3}{a-b}$.

Page 133. — **5.** $\frac{3ac^3}{5b}$. **6.** $\frac{3a^2}{4bx}$. **7.** $-\frac{13mqx}{17p}$. **8.** $\frac{a(a-b)}{3b}$. **9.** $\frac{4axy}{3x^2-8y}$. **10.** $\frac{2(a-b)}{9(a+b)}$. **11.** $\frac{x+2}{x-9}$. **12.** $\frac{x+12}{x(x-7)}$. **13.** $\frac{x+9}{4a(x-4)}$.

14. $\frac{x-6}{x^2-9x+20}$. 15. $\frac{x+y}{x^2+xy+y^2}$. 16. $\frac{x^2+12x+144}{x+11}$. 17. $\frac{x+a}{x+c}$.
18. $\frac{x-a}{x-c}$. 19. $\frac{x^2+y^2}{x^4+x^2y^2+y^4}$. 20. $\frac{a}{a^2+x^2}$. 21. $\frac{x^2}{x-a}$. 22. $\frac{a^2-ax+x^2}{a(a+x)}$.
23. $\frac{x+7}{4(x+6)}$. 24. $\frac{2x+5}{3x-6}$.

Page 134. — 25. $\frac{x^2-7x+2}{x^2-7x-5}$. 26. $\frac{2x-3}{7x+2}$. 27. $\frac{x+5}{x^2+5x+18}$.
28. $\frac{a^2+3x^2}{x(4a+3x)}$. 29. $\frac{x^2+3x+27}{x^2+3x+3}$. 30. $\frac{x^2-5x+25}{2x^2-3x+15}$. 31. $\frac{x^2+3x-7}{x^2+3x+16}$.
32. $\frac{x^2+4}{x^2+x+1}$. 33. $\frac{x^2-2x+3}{x^2+x-2}$. 34. $\frac{x-y}{x+y}$.

Page 135. — 3. $ax-b$. 4. $b+\frac{x}{a}$. 5. $b-\frac{x^2}{a}$. 6. x^2-y^2.
7. $x+\frac{y^2}{x+y}$. 8. x^2+xy+y^2. 9. $a+b+\frac{2b^2}{a-b}$. 10. $x-2-\frac{7}{x-1}$.
11. $a-x+\frac{3c}{a+x}$. 12. $x+2y-\frac{2y^2+5}{x+y}$. 13. $x^2-5ax+\frac{a^2}{3x}$. 14. $3+\frac{3}{x^2-1}$.
15. $a-b$. 16. $x^2+xy+y^2-\frac{z^3}{x-y}$.

Page 136. — 5. $\frac{3xy+x}{y}$. 6. $\frac{9ay^2-a^3}{y}$. 7. $\frac{2y}{x+y}$. 8. $\frac{2ax-4a}{x-1}$.
9. $\frac{2ax}{a+x}$. 10. $\frac{2a^3-ab+a^2b-2b^2}{a+b}$. 11. $\frac{x^3}{x+y}$. 12. $\frac{x^3-3x^2z+3xz^2}{x-z}$.
13. $\frac{xy}{x+y}$. 14. $\frac{2a-abx}{a+x}$.

Page 138. — 5. $\frac{a^2}{abc}, \frac{b^2}{abc}, \frac{c^2}{abc}$. 6. $\frac{4a^2}{12ac}, \frac{3b}{12ac}, \frac{2c^2}{12ac}$. 7. $\frac{6x}{10x^2}, \frac{25}{10x^2}, \frac{4x}{10x^2}$. 8. $\frac{4a^3-4a^2b}{12a^3}, \frac{3a^2+3ab}{12a^3}, \frac{2b^2}{12a^3}$. 9. $\frac{axy+5xy}{x^2y^2}, \frac{ay+6y}{x^2y^2}, \frac{ax+3x}{x^2y^2}$. 10. $\frac{x+1}{x^2-1}, \frac{1}{x^2-1}, \frac{x-1}{x^2-1}$. 11. $\frac{a^2-2ab+b^2}{a^2-b^2}, \frac{a+b}{a^2-b^2}, \frac{a^3+ab^2+a^2b+b^3}{a^2-b^2}$. 12. $\frac{ax+3a}{x^2+5x+6}, \frac{a^2x+2a^2}{x^2+5x+6}, \frac{2a}{x^2+5x+6}$.
13. $\frac{x^3+3x^2-3x-9}{x^2-9}, \frac{3x+3x^2-x^3-9}{x^2-9}, \frac{x^2}{x^2-9}$. 14. $\frac{a^2x^2+a^2y^2}{x^4-y^4}, \frac{b^2x^2-b^2y^2}{x^4-y^4}, \frac{a-b}{x^4-y^4}$.

Page 140. — 3. $\frac{13a}{5x}$. 4. $\frac{11x+2a}{2x^2}$. 5. $\frac{5-3x}{xy}$. 6. $\frac{x^2-2xy+y^2}{xy}$.
7. $\frac{4a^2x+5a^2-x^2}{4a^2x^2}$. 8. $\frac{abx+x^2}{aby}$. 9. $\frac{2x^2y^3-6y^4+6y^2+x^2y^2}{10x^2y^2}$.
10. $\frac{y^2-xy-1}{xy}$.

Page 141. — **12.** $\frac{2x}{x^2-y^2}$. **13.** $\frac{2b}{a^2-b^2}$. **14.** $\frac{2(x^2+y^2)}{x^2-y^2}$. **15.** $\frac{2a^4}{a^6-b^6}$.

16. $\frac{xy-2y-3x+9}{x^2-5x+6}$. **17.** $\frac{5x^2-13x+8}{x^2+x-12}$. **18.** $\frac{8a}{1-4a^2}$. **19.** $\frac{3(x-1)}{x^2-4}$.

20. $\frac{-2(7x+1)}{x^2-2x-15}$. **21.** $\frac{4abx}{a^2-b^2x^2}$. **22.** $\frac{1}{a-b}$. **23.** $\frac{ab(a+b)}{a^3-b^3}$. **24.** $\frac{2}{x-a}$.

25. $\frac{a^2+b^2}{a^2-b^2}$. **26.** 0. **27.** $\frac{x(a^2-2ax-x^2)}{a^2-x^2}$. **28.** $\frac{4x}{x+y}$. **29.** $\frac{-x}{x^2+ax+a^2}$.

30. $\frac{2a^2}{a^3+b^3}$. **31.** $\frac{y^2-xy}{x^4+x^2y^2+y^4}$. **32.** $\frac{x^2-a^2}{ax(x-y)}$. **33.** $\frac{a^3+a^2b-ax^2+bx^2}{ax(a^2-b^2)}$.

34. $\frac{a^2+ab-65a+60b}{a^2-b^2}$. **35.** $\frac{2xy^2}{x^4-y^4}$. **36.** 0. **37.** $\frac{2(x+1)}{x}$.

38. $\frac{2a^3+ab^2+2a^2b-b^3}{a^2b-b^3}$. **39.** 0. **40.** $\frac{m^3+mn^2+n^3}{(m+n)^3}$. **41.** $\frac{36}{(x+3)^2(x-3)}$.

Page 142. — **42.** $\frac{-x^2+4x-8}{x^2+2x-15}$. **43.** $\frac{x+3a}{x-3a}$. **44.** $\frac{1}{1-x}$.

45. $\frac{a^2+9ax}{(x+a)^2(a-x)}$. **46.** $\frac{8}{(x^2-16)(x-3)}$. **47.** $\frac{x(x^2-8x+8)}{(x^2-16)(x-5)(x-3)}$.

48. $\frac{x^2-4x-22}{x^3-10x^2+31x-30}$. **49.** $\frac{c-b}{(x+a)(x+b)(x+c)}$.

Page 143. — **51.** $\frac{a^2+x^2}{ax(a-b)}$. **52.** $\frac{6(a^2+b^2)}{a^2-b^2}$. **53.** $\frac{a^2-4ab-b^2}{a^4-2a^2b^2+b^4}$.

54. $\frac{1}{(a-c)(c-b)}$. **55.** $\frac{3c}{(x-a)(a-b)}$. **56.** 1. **57.** 0.

58. $\frac{x^2y+xy^2+2xz^2-2y^2z}{(x+y)(x-z)(y+z)}$. **59.** $\frac{3x^2-a^2-b^2-c^2}{(x-a)(x-b)(x-c)}$.

Page 144. — **3.** $\frac{3x}{b}$. **4.** $\frac{3}{x^2}$. **5.** $a-b$. **6.** $\frac{1}{a-b}$. **4.** $\frac{3ab^2}{10y^2}$.

5. $\frac{2b^2x^2}{5}$. **6.** $\frac{3x}{2}$. **7.** $a(x-y)$.

Page 145. — **8.** $\frac{1}{a+x}$. **9.** $\frac{a^2}{(a+b)^2}$. **10.** $\frac{x^2+xy+y^2}{x^2-xy+y^2}$. **11.** $\frac{x}{3}$.

12. $\frac{x+y}{x-y}$. **13.** $\frac{x+y}{x-y}$.

Page 146. — **4.** $\frac{3bx}{8ay}$. **5.** $\frac{a^4b^4}{2x^2y^2}$. **6.** $\frac{5x^2y^2}{2(x-y)}$. **7.** $\frac{a}{3x}$. **8.** $\frac{a^2}{a-b}$.

9. x^2-y^2. **10.** $\frac{2}{3(b-c)}$. **11.** xy. **12.** $x+1$. **13.** $\frac{a^3-b^3}{a+b}$. **1.** $\frac{2xy}{3}$.

2. $4abc$. **3.** $\frac{21x}{10a}$. **4.** $\frac{3ay}{2x}$. **5.** $\frac{35ac^2x^2y^2}{4b^2z}$. **6.** $\frac{6b^2}{a-b}$. **7.** $\frac{a-x}{a}$. **8.** $\frac{a}{x}$.

9. $\frac{3(a^3+ab^2+a^2b+b^3)}{5a}$. **10.** $\frac{2(x-y)^2}{5y}$.

Page 147. — 11. $\frac{6(a^2+ax+x^2)}{5(a-x)}$. 12. $\frac{x}{9(a^2-ax+x^2)}$. 13. $\frac{1}{a}$. 14. 1.
15. $\frac{a-b}{a^2b^2}$. 16. $\frac{(x-y)^2}{y}$. 17. $\frac{1}{x(x+1)}$. 18. $\frac{b^4-a^4}{b^3}$. 19. $\frac{1}{a^2-b^2}$.
20. $(x+y)^2$. 21. $\frac{1}{x^3-2x^2y+2xy^2-y^3}$. 22. $\frac{x+5}{x+4}$. 23. $\frac{x-4}{x-5}$. 24. 1.
25. $\frac{1}{x-7}$. 26. $\frac{(x^2-y^2)^2}{a}$. 27. $\frac{1}{a^2-x^2}$. 28. $\frac{ab}{a^2+b^2}$. 29. $\frac{p^2+q^2}{p^2-q^2}$.
30. x^2+y^2. 31. $(a^2x^2-1)^2$.

Page 149. — 4. $\frac{63}{80}$. 5. $\frac{x}{y}$. 6. $\frac{x}{x^2+x+1}$. 7. $\frac{x+a}{x-a}$. 8. $\frac{a+1}{ax+x-a}$.
9. $\frac{b+c}{a}$. 10. $a-1$. 11. $\frac{b(a-x)}{a(b-x)}$. 12. x. 13. $\frac{1}{1+a}$. 14. $\frac{a+b}{b}$.
15. $\frac{x^2-a^2}{2x}$. 16. $\frac{a(b^2-1)}{b(a^2-1)}$. 17. $\frac{1-2a-a^2}{1-3a}$. 18. $a-1$. 19. $\frac{x^2+2xy}{y^2}$.

Page 150. — 1. $\frac{a^2+b^2}{a^2-b^2}$. 2. $\frac{a^2+a+1}{a-1}$. 3. $\frac{a^2}{a^2+1}$. 4. $\frac{a^2(a-b)}{a+b}$.
5. $\frac{a^2+1}{a^4+a^2+1}$. 6. $\frac{a^2+x^2}{a^2+ax+x^2}$. 7. $3a+2b$. 8. $\frac{x+5}{x+3}$. 9. $\frac{x+4}{2x+3}$.
10. $-\frac{ab}{a-b}$. 11. $\frac{1}{a-x}$. 12. $\frac{1}{x}$. 13. $\frac{a}{a+1}$. 14. $\frac{x^2}{x^2-a^2}$. 15. $\frac{2}{a(a^2-1)}$.
16. $\frac{5a-2}{ay}$. 17. $\frac{8xy}{x^2-4y^2}$. 18. $-\frac{7}{x^2-1}$. 19. $\frac{x^3y+x^2y^2+xy^3-y^4}{x^4-y^4}$.
20. $\frac{15x^2+39ax-24a^2}{20a(a+x)}$. 21. $\frac{8a^2x-4ax^2+4a^3-a^2+x^2}{4a^2x^2}$. 22. $\frac{x^2}{y^2}$.
23. $\frac{b^2-3ab}{a^2-b^2}$. 24. $\frac{6x-6}{x^2-9}$. 25. $\frac{a}{b}$. 26. $a(a^2+b^2)$. 27. $\frac{x-1}{x-5}$.
28. $\frac{x^2(x+y)}{a-b}$.

Page 151. — 29. $\frac{2y+x}{10}$. 30. $\frac{abx-xy+x^2}{aby}$. 31. $\frac{xy^2+xz^2+y^2z}{xyz}$.
32. $\frac{xy}{x^2-y^2}$. 33. $\frac{15(x+4)}{x}$. 34. $\frac{1-m}{m^2-m^3-m}$. 35. $\frac{2n^2}{m^2+3mn+2n^2}$.
36. $\frac{m^2+n^2}{2mn}$. 37. $\frac{2ab}{a^2+b^2}$. 38. $\frac{ax-by}{ay+bx}$. 39. $\frac{y(a-1)-x(a+1)}{y(a+1)-x(a-1)}$. 40. $\frac{m^2+n^2}{2m^2}$.
41. 1. 42. $\frac{2a^4}{a^4+x^4}$. 43. $\frac{1-a+a^2}{1+a}$. 44. $\frac{2xy}{y^2+2xy-x^2}$.

Page 153. — 5. $x=50$. 6. $x=3\frac{3}{16}$. 7. $x=6$. 8. $x=36$. 9. $x=12$.
10. $x=6$. 11. $x=7$. 12. $x=9$. 13. $x=12$. 14. $x=9$. 15. $x=5$.
16. $x=10$. 17. $x=4$. 18. $x=6$. 19. $x=8$. 20. $x=1$. 21. $x=1$.
22. $x=1$. 23. $x=10$.

Page 154. — **24.** $x = 2\frac{4}{5}$. **25.** $x = 2\frac{2}{5}$. **26.** $x = 3\frac{5}{8}$. **27.** $x = 1\frac{1}{2}$. **28.** $x = -4\frac{1}{2}$. **29.** $x = -2$. **30.** $x = 0$. **31.** $x = -2$. **32.** $x = 5$. **33.** $x = 1\frac{1}{2}$. **34.** $y = 5\frac{5}{9}$. **35.** $x = 2$. **36.** $x = 4$. **37.** $x = 8$. **38.** $x = 7$. **39.** $x = 1$. **40.** $x = 2$. **41.** $x = 4$. **42.** $x = 1\frac{2}{3}\frac{1}{9}$. **43.** $x = 4$. **44.** $x = 0$. **45.** $x = \frac{6}{7}$. **46.** $x = 2$.

Page 155. — **3.** $x = \frac{c}{a-b}$. **4.** $x = \frac{6c}{3a+2b}$. **5.** $\frac{b(m+d+c)}{a}$. **6.** $x = \frac{a(b-d)}{a-c}$. **7.** $x = \frac{a^2-b^2}{2b}$. **8.** $x = \frac{am(b+n)}{m-a}$. **9.** $x = \frac{(a^2-b^2)(c+d)}{4ab}$. **10.** $x = \frac{mn-pr}{m+n+p+r}$. **11.** $x = a - b$.

Page 156. — **12.** $x = a + b$. **13.** $x = -\frac{5}{6a+2b}$. **14.** $x = \frac{b-ab-a^2}{a-b}$. **15.** $x = \frac{b}{b-d}$. **16.** $x = \frac{a-1}{b}$. **17.** $x = \frac{2ab}{2b-a}$. **18.** $x = \frac{c^2-ab}{a+b-2c}$. **19.** $x = \frac{a^2+ab+b^2}{a+b}$. **20.** $x = \frac{m-n}{m+n}$. **21.** $x = \frac{m}{2n}$. **22.** $x = pqr$.

1. 72 and 60. **2.** $\frac{an}{n+m}$ and $\frac{am}{n+m}$. **3.** 24, 36, and 48.

Page 157. — **5.** $\frac{m}{6}$, $\frac{m}{3}$, and $\frac{m}{2}$. **6.** First day, 27 miles; second, 36 miles; third, 24 miles. **7.** Eldest, \$3000; second, \$2400; youngest, \$1800. **8.** First, \$280; second, \$210; third, \$245. **9.** Carriage, \$88; horse, \$187. **10.** Horse, \$120; saddle, \$20; bridle, \$10. **11.** B's age, 50 yrs.; A's, 40 yrs. **12.** Husband, 30 yrs.; wife, 18 yrs. **13.** $3\frac{1}{13}$ days.

Page 158. — **15.** $4\frac{4}{5}$ days. **16.** $1\frac{3}{7}$ days. **17.** $6\frac{1}{3}\frac{8}{7}$ days. (2) A, $14\frac{2}{1}\frac{2}{7}$ days; B, $18\frac{6}{13}$ days; C, $34\frac{2}{7}$ days. **18.** 24 days. **19.** 24 days. **20.** \$56. **21.** First, \$3258; second, \$2896.

Page 159. — **22.** Smaller, \$3240; larger, \$4860. **23.** First, 50 yards; second, 45 yards; third, 40 yards. **24.** $11\frac{1}{9}$ hours. **25.** 20 days. **26.** 60 sheep. **27.** 36 persons, 50¢; 18 persons, 75¢; 46 persons, \$1.50. **28.** \$1000. **29.** Longest side, 36 rods; second, 27 rods; third, 18 rods. **30.** 40 eggs.

Page 160. — **31.** 10 and 11. **32.** Men, \$12; women, \$9; children, \$4.50. **33.** B, \$1850; A, \$2150; C \$2000. **34.** C, $\frac{m+3n}{3}$; A, $\frac{m}{3}$; B, $\frac{m-3n}{3}$. **35.** 540 yards. **37.** \$45 and \$75. **38.** 75 and 50. **39.** $\frac{am}{a-b}$ and $\frac{bm}{a-b}$.

Page 161. — **40.** 60 apples. **41.** Pears, 40; lemons, 28; oranges, 8. **42.** 8 children. **43.** Barley, 18 bushels; corn, 24 bushels; oats, 30 bushels. **44.** 1059 men. **45.** Five-cent stamps, 9; two-cent, 15; one-cent, 25. **46.** 30 acres. **47.** 21 days.

Page 162. — **48.** $\frac{n+bm}{a+b}$ days. **49.** $4\frac{4}{9}$ hours. **50.** C, 40 days; A, 120 days; B, 60 days. **51.** 50 yrs. **52.** (1) $10\frac{10}{11}$ minutes past 2; (2) $21\frac{9}{11}$ minutes past 4. **53.** (1) $49\frac{1}{11}$ minutes past 3; (2) $10\frac{10}{11}$ minutes past 8. **54.** $1000. **55.** 36 miles. **56.** $6\frac{2}{3}$ hours. **57.** $14\frac{2}{7}$ miles.

Page 163. — **3.** 4 P.M.

Page 164. — **4.** 7.30 A.M.

Page 165. — **8.** First, 350 miles; second, 190 miles. **10.** 12, 18, 5, and 45.

Page 166. — **11.** 12, 28, and 60. **12.** $\frac{m}{2+n}-n$, $\frac{m}{2+n}+n$, and $\frac{mn}{2+n}$.

Page 167. — **16.** 187 pounds. **17.** 42%. **18.** 4000 pounds.

Page 168. — **19.** $28.94. **20.** 8%. **21.** 3 yr. 11+mo. **22.** $620. **23.** 3 yr. 9 mo. **24.** $280.74+.

Page 172. — **5.** $x=13$; $y=5$. **6.** $x=8$; $y=6$. **7.** $x=7$; $y=5$. **8.** $x=10$; $y=6$. **9.** $x=24$; $y=4$. **10.** $x=11$; $y=5$. **11.** $x=8$; $y=5$. **12.** $x=8$; $y=6$. **13.** $x=12$; $y=18$. **14.** $x=50$; $y=20$. **15.** $x=16$; $y=12$. **16.** $x=3$; $y=2$.

Page 174. — **19.** $x=5$; $y=4$. **20.** $x=3$; $y=7$. **21.** $x=7$; $y=1$. **22.** $x=5$; $y=2$. **23.** $y=14$; $z=10$. **24.** $x=11$; $y=7$. **25.** $x=4$; $y=2$. **26.** $x=6$; $z=12$. **27.** $x=10$; $y=9$. **28.** $x=6$; $y=4$.

Page 175. — **31.** $x=12$; $y=8$. **32.** $x=2$; $z=3$. **33.** $y=4$; $z=5$. **34.** $x=3$; $z=2$. **35.** $y=5$; $z=1$. **36.** $x=\frac{1}{2}$; $y=\frac{1}{2}$. **37.** $y=14$; $z=10$. **38.** $x=11$; $z=13$. **39.** $x=16$; $y=12$. **40.** $x=7$; $z=3$. **41.** $x=12$; $y=6$. **42.** $x=\frac{18}{a}$; $y=\frac{8}{b}$. **43.** $x=16$; $y=12$.

Page 176. — **1.** $x=13$; $y=6$. **2.** $x=2$; $y=3$. **3.** $x=11$; $y=2$. **4.** $x=8$; $y=4$. **5.** $x=4$; $y=3$. **6.** $x=10$; $y=5$.

Page 177. — **7.** $x=5$; $y=6$. **8.** $x=8$; $y=10$. **9.** $x=12$; $y=9$. **10.** $x=-1$; $y=2$. **11.** $x=6$; $y=12$. **12.** $x=-\frac{2}{3}$; $y=\frac{10}{9}$. **13.** $x=3$; $y=2$. **14.** $x=\frac{1}{2}$; $y=\frac{1}{4}$. **15.** $x=7$; $y=3$. **16.** $x=10$; $y=2$. **17.** $x=38\frac{1}{2}$; $y=70$. **18.** $x=6$; $y=12$. **19.** $x=11\frac{1}{9}$; $y=4\frac{4}{9}$. **20.** $x=3$; $y=5$. **21.** $x=9$; $y=123\frac{1}{4}$. **22.** $x=5\frac{1}{2}$; $y=2\frac{1}{10}$. **23.** $x=5$; $y=8$. **24.** $x=6$; $y=4$. **25.** $x=13.1$; $y=5.6$. **26.** $x=5$; $y=3$. **27.** $x=4$; $y=5$. **28.** $x=8$; $y=2$. **29.** $x=\frac{ab(3a-4b)}{4(a^2-b^2)}$; $y=\frac{ab(4a-3b)}{4(a^2-b^2)}$. **30.** $x=\frac{s+d}{2}$; $y=\frac{s-d}{2}$.

Page 178. — **31.** $x=\frac{bd+c}{a+b}$; $y=\frac{ad-c}{a+b}$. **32.** $\frac{m(s-cn^2)}{m^2-n^2}$; $\frac{n(cm^2-s)}{m^2-n^2}$. **33.** $x=\frac{mn(an+bm)}{m^2+n^2}$; $y=\frac{mn(am-bn)}{m^2+n^2}$. **34.** $x=\frac{11(a+b)}{2}$; $y=\frac{9(a-b)}{2}$. **35.** $x=\frac{m^2n^2-mn^3-mn-m^2}{m^2n^2-1}$; $y=\frac{m^3n+m^2n^2-mn+n^2}{m^2n^2-1}$. **36.** $x=\frac{ab^2c}{a^2+b^2}$; $y=\frac{a^2bc}{a^2+b^2}$. **37.** $x=\frac{a(c-2b)}{a-b}$; $y=\frac{b(2a-c)}{a-b}$. **38.** $x=\frac{3m}{2}$; $y=\frac{n}{2}$.

Page 179. — **40.** $x=4$; $y=6$. **41.** $x=4$; $y=6$. **42.** $x=3$; $y=5$. **43.** $x=\frac{n^2-m^2}{bn-am}$; $y=\frac{n^2-m^2}{an-bm}$. **44.** $x=3$; $y=2$. **45.** $x=7$; $y=6$. **46.** $x=\frac{n^2-m^2}{nq-mp}$; $y=\frac{n^2-m^2}{np-mq}$. **47.** $x=\frac{2}{a+b}$; $y=\frac{2}{a-b}$.

Page 181. — **5.** $x=3$; $y=2$; $z=1$. **6.** $x=7$; $y=5$; $z=4$. **7.** $x=3$; $y=4$; $z=6$. **8.** $x=2$; $y=3$; $z=4$. **9.** $x=2$; $y=3$; $z=7$. **10.** $x=2$; $y=3$; $z=5$.

Page 182. — **11.** $x=5$; $y=4$; $z=3$. **12.** $x=8$; $y=4$; $z=2$. **13.** $x=11$; $y=9$; $z=4$. **14.** $x=6$; $y=12$; $z=20$. **15.** $x=24$; $y=60$; $z=120$. **16.** $x=3$; $y=4$; $z=6$. **17.** $x=7\frac{1}{8}$; $y=7$; $z=-1\frac{7}{8}$. **18.** $x=y=z=\frac{abc}{ab+ac+bc}$. **19.** $x=\frac{2}{a+b-c}$; $y=\frac{2}{a-b+c}$; $z=\frac{2}{b-a+c}$. **20.** $x=4$; $y=9$; $z=16$; $v=25$. **1.** 147 and 196. **2.** 65 and 24. **3.** Boy's age, 18 yrs.; girl's, 7 yrs.

Page 183. — **4.** A's age, $\frac{an+bmn}{m-n}$; B's age, $\frac{a+bn}{m-n}$. **5.** First, \$2500; second, \$7500. **6.** Silver, 552 ounces; copper, 60 ounces. **7.** Nickel, 20 ounces; copper, 28 ounces; silver, 12 ounces. **8.** 24. **9.** A, \$300; B, \$500. **10.** Tea, 85 cents; sugar, 8 cents. **11.** Tea, $\frac{dm-bn}{ad-bc}$; sugar, $\frac{an-cm}{ad-bc}$. **12.** $\frac{5}{8}$.

Page 184. — **13.** $\frac{13}{17}$. **14.** $\frac{17}{20}$. **15.** 1 and $6\frac{2}{3}$. **16.** Upper, 45 inches; lower, 63 inches. **17.** Finer, \$1.20; coarser, \$.80. **18.** 10 and 2. **19.** A's age, 49 yrs.; B's age, 21 yrs. **20.** Persons, 13; sum, \$3.

Page 185. — **21.** Larger, $10\frac{1}{2}$ hours; smaller, 14 hours. **22.** 9, 15, and 24. **23.** 276. **24.** A, \$3$\frac{1}{4}$; B, \$3; C, \$2$\frac{1}{4}$. **25.** First, \$8; second, \$18; third, \$16. **26.** Wheat, \$1.25; rye, \$.95; oats, \$.80. **27.** 2, 5, 7, and 10. **28.** First horse, \$56; second, \$33.

Page 187. — **3.** y^{12}. **4.** y^{4n}. **5.** x^{nr}. **3.** $243\,a^{10}x^5y^{15}$. **4.** $\frac{a^4}{x^8}$. **5.** $\frac{x^{n^2}}{y^{mn}}$.

Page 188. — **6.** $64\,a^3b^6$. **7.** $-\frac{64}{27\,a^3x^3}$. **8.** $27\,a^6b^3$. **9.** $25\,a^4x^2y^6$. **10.** $-64\,a^3b^6x^3$. **11.** $-a^{10}b^5c^5x^{10}$. **12.** $729\,a^{18}b^{12}c^6$. **13.** $-8\,a^9x^3y^6$. **14.** $81\,a^4b^2x^{\frac{8}{3}}y^{12}$. **15.** $a^6b^6c^4$. **16.** $a^{2n}b^{2m}c^2$. **17.** $2^n a^{mn}x^{n^2}y^{nr}$, when n is even; $-2^n a^{mn}x^{n^2}y^{nr}$, when n is odd. **18.** $3^{2n}a^{2n}b^{2n}x^{2n}$. **19.** $x^{2ny}z^{2n}$. **20.** $-a^{2n+1}b^{2n+1}c^{2n+1}$. **21.** $-27\,x^{3my^{6m}}$. **22.** $\frac{9\,a^4x^2}{16\,y^2}$. **23.** $-\frac{8\,x^6y^9}{125\,a^3b^6}$. **24.** $\frac{625\,x^4y^4}{81\,z^8}$. **25.** $\frac{a^nb^nc^n}{x^n}$. **26.** $\frac{a^{2m}b^{2m}}{x^my^m}$. **27.** $\frac{a^{2n}}{x^{2n}y^{2n}}$. **28.** $-\frac{x^{2n+1}}{a^{2n+1}}$.

Page 190. — **1.** $x^5-5\,x^4y+10\,x^3y^2-10\,x^2y^3+5\,xy^4-y^5$.

Page 191. — **2.** $a^6-6\,a^5b+15\,a^4b^2-20\,a^3b^3+15\,a^2b^4-6\,ab^5+b^6$. **3.** $m^4-4\,m^3n+6\,m^2n^2-4\,mn^3+n^4$. **4.** $a^5+5\,a^4x+10\,a^3x^2+10\,a^2x^3+5\,ax^4+x^5$. **5.** $a^5-5\,a^4x+10\,a^3x^2-10\,a^2x^3+5\,ax^4-x^5$. **6.** $x^7+7\,x^6y+21\,x^5y^2+35\,x^4y^3+35\,x^3y^4+21\,x^2y^5+7\,xy^6+y^7$. **7.** $c^4-4\,c^3d+6\,c^2d^2-4\,cd^3+d^4$. **8.** $a^6+6\,a^5b+15\,a^4b^2+20\,a^3b^3+15\,a^2b^4+6\,ab^5+b^6$.

9. $1-3x+3x^2-x^3$. **10.** $x^4-4x^3+6x^2-4x+1$. **11.** $-56a^5b^3$. **12.** $120c^7d^8$. **13.** $-84x^6y^3$. **14.** $a^8-8a^7b+28a^6b^2-\cdots$. **15.** $c^9+9c^8d+36c^7d^2+\cdots$. **16.** $x^{10}-10x^9y+45x^8y^2-\cdots$. **17.** $\cdots+15m^2n^4-6mn^5+n^6$. **18.** $\cdots+28a^2x^6+8ax^7+x^8$. **19.** $\cdots-21a^2b^5+7ab^6-b^7$.

Page 192. — **21.** $27a^3-54a^2b^2+36ab^4-8b^6$. **22.** $a^6-9a^4bc+27a^2b^2c^2-27b^3c^3$. **23.** $a^4b^4-2a^3b^3c+\frac{3}{2}a^2b^2c^2-\frac{1}{2}abc^3+\frac{c^4}{16}$. **24.** $\frac{a^4}{16}-\frac{1}{2}a^3bc+\frac{3}{2}a^2b^2c^2-2ab^3c^3+b^4c^4$. **25.** $32m^5-80m^4n^3+80m^3n^6-40m^2n^9+10mn^{12}-n^{15}$. **26.** $\frac{a^3b^3}{8}-3a^2b^2c+24abc^2-64c^3$.

Page 193. — **3.** $25a^4x^2y^6$. **4.** $216a^3b^6c^9x^{12}$. **5.** $-32a^{10}x^{15}y^{20}$. **6.** $81m^{12}n^4x^8y^4$. **7.** $-p^{14}x^7y^{21}$. **8.** $d^6x^{12}y^6z^{18}$. **9.** $\frac{a^4}{m^6}$. **10.** $\frac{c^8x^{12}}{81n^4y^4}$. **11.** $-\frac{8a^3c^9}{8m^3n^3}$. **12.** $-\frac{x^{10}y^{10}z^5}{32a^5b^{15}c^5}$. **13.** $\frac{a^nb^{3n}c^n}{x^ny^{2n}z^n}$, when n is even; $-\frac{a^nb^{3n}c^n}{x^ny^{2n}z^n}$, when n is odd. **14.** $\frac{x^{m^2}y^{mn}}{a^{m^2}c^{mn}}$. **15.** $m^4+4m^3n+6m^2n^2+4mn^3+n^4$. **16.** $m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$. **17.** $a^6-3a^4b+3a^2b^2-b^3$. **18.** $x^4-4x^3+6x^2-4x+1$. **19.** $1-3x+3x^2-x^3$. **20.** $1-4z^2+6z^4-4z^6+z^8$. **21.** $x^9-9x^6a^2+27x^3a^4-27a^6$. **22.** $y^{12}-9y^8z^2+27y^4z^4-27z^6$. **23.** $16a^4+16a^3x+6a^2x^2+ax^3+\frac{x^4}{16}$. **24.** $\frac{a^3b^3}{27}-\frac{a^2b^2c^2}{3}+abc^4-c^6$. **25.** $x^4-4x^2a+6a^2-\frac{4a^3}{x^2}+\frac{a^4}{x^4}$. **26.** $x^6+\frac{3x^5}{a^2}+\frac{3x^4}{a^4}+\frac{x^3}{a^6}$. **27.** $x^{15}+15x^{14}y+105x^{13}y^2+\cdots+105x^2y^{13}+15xy^{14}+y^{15}$. **28.** $x^{10}-10x^9a^2+45x^8a^4-\cdots+45x^2a^{16}-10xa^{18}+a^{20}$. **29.** $x^{12}+12x^{11}y+66x^{10}y^2+\cdots+66x^2y^{10}+12xy^{11}+y^{12}$. **30.** $-792a^7c^5$. **31.** $-252x^5$. **32.** $-126x^4$.

Page 194. — **1.** $5ax^2$. **2.** $2x^2yz^3$. **3.** $4m^2nx^3$.

Page 195. — **4.** $\pm5ab$. **5.** $\pm9a^2b^3c^4$. **6.** $\pm4x^2y$. **7.** $4ab^2c^3$. **8.** $-3mn^2$. **9.** $\pm2ab^2x^4$. **10.** $-5mn^2s^4$. **11.** $\pm3x^2y^3z$. **12.** $-3a^2c^4$. **13.** $-2a^2x$. **14.** ab. **15.** a^mx^2. **16.** $\pm x^ny^m$. **17.** x^4y^3. **18.** $\pm\frac{4a^2b}{5xy^2}$. **19.** $-\frac{6ax^2}{5b^2c^4}$. **20.** $-\frac{2ab^2}{x^2y}$. **21.** $\pm\frac{3a^2x}{4m^3n^4}$. **22.** $\frac{a^mx^3}{y^2z}$. **23.** $\frac{x^my^2}{z^s}$. **24.** $-\frac{x^my^{2n}}{a^2b^m}$.

Page 196. — **1.** $x-2y$. **2.** $2x^2+y$. **3.** x^2-3y^2. **4.** $3a-2y^2$. **5.** $2a-1$. **6.** $1+3y^3$. **7.** $3x-\frac{y}{3}$. **8.** $\frac{x}{2}-y^2$. **9.** $x+y-2$. **10.** $x^2+\frac{2}{3}y^2+3z+4$.

Page 197. — **12.** $x^2-2xy+y^2$. **13.** $3x^2-2xy+5y^2$. **14.** a^2-5a-5. **15.** $1-5x+x^2$. **16.** $2x-xy+\frac{1}{2}y$.

Page 198. — **17.** $1-2x+3y$. **18.** $9-a+b$. **19.** $\frac{1}{2}x^2+\frac{2}{3}y+\frac{3}{4}$. **20.** $\frac{2}{3}x-\frac{3}{2}y$. **21.** $x-y+\frac{2}{3}$. **22.** $1+x+y+z$. **23.** $1-x+x^2-x^3+x^4$. **24.** $a^3-3a+\frac{3}{a}-\frac{1}{a^3}$.

Page 202. — **14.** 263. **15.** 307. **16.** 240. **17.** 459. **18.** 702. **19.** 723. **20.** 625. **21.** 13.3. **22.** 6.72. **23.** .094. **24.** .025. **25.** 8.68+. **26.** 24.22+. **27.** .238. **28.** $\frac{8}{3}$. **29.** $16\frac{1}{2}$. **30.** $32\frac{1}{4}$. **31.** .591+. **32.** .73+. **33.** $\frac{4}{3}\frac{5}{3}$. **34.** $\frac{5}{1}\frac{2}{4}$. **35.** 1.414+. **36.** 1.732+. **37.** 2.236+. **38.** 1.870+. **39.** 2.569+. **40.** .948+. **41.** .774+. **42.** .816+. **43.** .935+. **44.** .845+. **45.** .516+. **46.** .741+.

Page 203. — **3.** $x^2 + 2a$. **4.** $2x - a$. **5.** $5a^3 - 3b^2$. **6.** $a^2 + a + 1$. **7.** $x^2 + 2x - 4$.

Page 204. — **8.** $2x^2 - x - 5$. **9.** $1 + 3x - 3x^2$. **10.** $x - \frac{2}{x}$. **11.** $a - 1 + \frac{2}{a}$. **12.** $3a^2 - 4ab - 2b^2$. **13.** $2x^2 - 3xy + y^2$. **14.** $3x^2 - 2ax + a^2$.

Page 206. — **8.** 551368. **9.** 314432. **10.** 456533. **11.** 175616. **12.** 1124864. **13.** 1953125.

Page 207. — **16.** 35. **17.** 57. **18.** 145. **19.** 364. **20.** 325. **21.** 301. **22.** 4.2. **23.** 4.6. **24.** .46. **25.** 12.5. **26.** 4.07. **27.** 3.04. **28.** $\frac{1}{2}$. **29.** $\frac{1}{1}\frac{3}{5}$. **30.** $1\frac{1}{5}$. **31.** $3\frac{1}{3}$. **32.** .712+. **33.** 1.908 nearly. **34.** 4. **35.** 1.259+. **36.** 2.08+. **37.** 2.15+. **38.** .873+. **39.** .941+. **40.** .793+. **41.** .427+. **42.** .464+. **43.** .411+.

Page 208. — **44.** $2a + 3b$. **45.** $x^2 - 2x - 2$. **46.** $x + y$. **47.** $1 + 2a$.

Page 211. — **1.** $\sqrt[3]{a^3b^3}$. **2.** $\sqrt{(x-y)^2}$. **3.** $\sqrt[5]{\frac{243}{x^{10}}}$. **4.** $\sqrt[6]{a^3x^3}$. **5.** $\sqrt[6]{(x+y)^2}$. **6.** $\sqrt[n]{(a-x)^{\frac{n^2}{2}}}$. **7.** $\sqrt[m]{a^{\frac{m}{n}}b^{\frac{m}{n}}}$. **8.** $\sqrt{ab^2}$. **9.** $\sqrt[n]{x-y}$. **10.** $\sqrt{\frac{3}{x+y}}$. **11.** $\sqrt[4]{\frac{a^4}{x^4}}$. **12.** $\sqrt[6]{\frac{x^3}{(x-y)^3}}$. **13.** $\sqrt[n]{acd^2}$. **14.** $\sqrt{\frac{a+b}{ab}}$.

Page 212. — **15.** $\sqrt[12]{15625}$, $\sqrt[12]{256}$, $\sqrt[12]{1000}$. **16.** $\sqrt[12]{x^8}$, $\sqrt[12]{x^3y^3}$, $\sqrt[12]{x^6y^6}$. **17.** $\sqrt[12]{a^3}$, $\sqrt[12]{4b^2}$, $\sqrt[12]{a^3b^6}$. **18.** $\sqrt[12]{\frac{m^4}{n^4}}$, $\sqrt[12]{\frac{8}{n^3}}$, $\sqrt[12]{\frac{m^2}{9}}$. **19.** $\sqrt{\frac{1}{a}}$, $\sqrt{\frac{1}{b}}$, $\sqrt{\frac{1}{c}}$. **20.** $\sqrt[6]{x^6}$, $\sqrt[6]{x^4y^2}$, $\sqrt[6]{x^3y^3}$. **21.** $\sqrt[12]{(x-y)^3}$, $\sqrt[12]{(x+y)^2}$. **22.** $\sqrt{(a-b)^2}$, $\sqrt{a-b}$, $\sqrt{a-b}$. **23.** $\sqrt[10]{a^{10}b^5}$, $\sqrt[10]{\frac{b^2}{a^4}}$, $\sqrt[10]{\frac{a^2b^2}{(a+b)^5}}$. **1.** $5\sqrt{3}$. **2.** $4\sqrt[3]{5}$. **3.** $4\sqrt[4]{5}$. **4.** $3\sqrt[3]{3}$. **5.** $4ab^2\sqrt{2a}$. **6.** $15ab\sqrt[3]{5by^2}$.

Page 213. — **7.** $5a\sqrt{5a-2b}$. **8.** $2\sqrt[3]{2x^6y^2-3x^2y^6}$. **9.** $(a+b)\sqrt{a-b}$. **10.** $(x-y)\sqrt{x^2+xy+y^2}$. **11.** $(x+y)\sqrt[3]{x-y}$. **12.** $(x^2-y^2)\sqrt{x}$. **13.** $\frac{2b}{5}(a^2-25)\sqrt{3}$. **14.** $5a(1-y)\sqrt{x}$. **15.** $(x^2+xy)\sqrt{x^2-y^2}$. **16.** $\frac{1}{6}\sqrt{21}$. **17.** $\frac{1}{2}\sqrt[3]{10}$. **18.** $\frac{1}{3}\sqrt[4]{18}$. **19.** $\sqrt[3]{18}$. **20.** $2a\sqrt{15x}$. **21.** $2a\sqrt[3]{15ab}$. **22.** $\frac{1}{a}\sqrt{ax}$. **23.** $\frac{1}{a^2-b^2}\sqrt{5(a^2-b^2)}$. **24.** $\frac{2a}{a-b}\sqrt{3(a^2-b^2)}$. **25.** $\frac{1}{x-y}\sqrt{x^2-y^2}$. **26.** $(x+y)\sqrt[3]{x-y}$. **27.** $\sqrt[3]{x^2-1}$.

Page 214. — **28.** $\sqrt{10}$. **29.** $\sqrt{5ax}$. **30.** $\sqrt{a^2-b^2}$. **31.** $(a+b)\sqrt{a^2-b^2}$. **32.** $\sqrt[3]{(a+b)(a-b)^2}$. **33.** $\sqrt[3]{2(a-3b)}$. **34.** $y\sqrt{x(x-y)}$. **35.** $\sqrt[4]{a^2b(a+b)}$. **1.** $\sqrt[3]{8ax}$. **2.** $\sqrt{90a^2b}$. **3.** $\sqrt{\frac{5xy}{4}}$. **4.** $\sqrt{\frac{a^3y^2}{4}}$. **5.** $\sqrt[3]{2ax}$. **6.** $\sqrt[4]{9x^2y}$.

7. $\sqrt{m^2-1}$. 8. $\sqrt{\frac{m+n}{m-n}}$. 9. $\sqrt[3]{\frac{m^2+mn+n^2}{(m-n)^2}}$. 10. $\sqrt{\frac{a^2-b^2}{a^2-ab+b^2}}$.
11. $\sqrt[3]{a-b}$. 12. $\sqrt{1-a^2}$. 13. $\sqrt{-(a-b)b}$.

Page 215. — 3. $7\sqrt{2}$. 4. $17\sqrt{6}$. 5. $15\sqrt{5}$. 6. $-\frac{8}{15}\sqrt{30}$. 7. $\frac{13}{60}\sqrt{15}$.
8. $19\sqrt[4]{2}$. 9. $7\sqrt[3]{5}$. 10. $20\sqrt{5}$. 11. $-\frac{1}{3}\sqrt{2}$. 12. $\frac{9}{4}\sqrt{2}$.

Page 216. — 13. $1\frac{7}{8}\sqrt{3}$. 14. $1\frac{3}{8}\sqrt[3]{2}$. 15. $\frac{19}{3}\sqrt[4]{3}$. 16. $-3\sqrt[3]{10}$.
17. $\frac{3}{8}\sqrt[3]{4}$. 18. $\frac{1}{a}\sqrt{a}$. 19. $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{x}\right)\sqrt{3}$. 20. $\frac{3}{4}\sqrt{2}$. 21. $ay(3x-5y)\sqrt{x}$.
22. 0. 23. $3(a^3-x^3)\sqrt[3]{a^2x^2}$. 24. $\frac{2cy+30ax}{ay^2}\sqrt{xy}$. 25. 0.
26. $(5a-5x-2ax)\sqrt{a-x}$.

Page 217. — 4. 168. 5. $50\sqrt{6}$. 6. $\frac{3}{8}\sqrt[3]{3}$. 7. $\frac{2}{3}\sqrt[4]{5}$. 8. $\frac{3}{2}\sqrt{6}$. 9. $\frac{1}{6}\sqrt{15}$.
10. $120\sqrt[3]{4}$. 11. $12\sqrt[3]{18}$. 12. $\sqrt[6]{337500}$. 13. $6\sqrt[6]{54}$. 14. $24\sqrt[3]{6}$. 15. $\sqrt[6]{18}$.
16. $6\sqrt[13]{\frac{1}{17496}}$. 17. $12ab\sqrt[4]{a}$. 18. $6x\sqrt{5ay}$. 19. $18ax^2\sqrt{5x}$. 20. $6a^2xy$.
21. $\frac{3ayz}{10}\sqrt{30}$. 22. $6ab^2\sqrt[6]{a^4b^3}$. 23. $12\sqrt[6]{\frac{(a+x)^3}{a-x}}$. 24. $abcx^2\sqrt[3]{x}$.
25. $ab^2\sqrt[6]{864a^5b^2}$. 26. $15ab\sqrt[6]{x^2-y^2}$. 27. $-5\sqrt[3]{10}$.

Page 218. — 29. $x-\sqrt{x}-6$. 30. $6x-8\sqrt{3x}-24$. 31. $6+\sqrt{10}$.
32. $2a-\sqrt{ab}-3b$. 33. $16a+8\sqrt{ab}+b$. 34. $x\sqrt{x}-y\sqrt{y}$. 35. $a^2\sqrt{3}$
$+3a^2-a-1$. 36. $2\sqrt[3]{9}+5\sqrt[3]{6}+2\sqrt[3]{4}$. 37. $\sqrt{2}+\frac{3}{2}\sqrt{6}-\frac{1}{3}\sqrt{3}-3$.
38. $4a-b$. 39. $x-4y$. 40. 33. 41. $\frac{1}{4}x-12$. 42. $\sqrt{x^2-21}$.
43. $-x$. 44. $2x$.

Page 219. — 3. $6\sqrt{2}$. 4. $\frac{3}{15}\sqrt{6}$. 5. $\frac{3}{8}\sqrt{3}$. 6. $\frac{3}{4}\sqrt{3}$. 7. $\frac{8}{3}$. 8. $\frac{6}{5}$.
9. 2. 10. $\frac{1}{3}\sqrt[3]{36}$. 11. $\frac{1}{2}\sqrt[6]{18}$. 12. $\frac{1}{3}\sqrt[6]{4320}$. 13. $\sqrt[6]{40}$. 14. $2\sqrt[6]{5}$. 15. $5a\sqrt{b}$.
16. $\frac{2a^2}{5x^2}\sqrt[6]{ax^3}$. 17. $\frac{a}{2x}\sqrt[6]{ax}$. 18. $\frac{5}{4x}\sqrt[6]{x^5}$. 19. $\frac{3a^2}{2x^2}\sqrt[6]{a^3x^5}$. 20. $\sqrt{x+y}$.

Page 220. — 1. $\frac{5}{3}\sqrt{3}$. 2. $\frac{1}{3}\sqrt{15}$. 3. $5\sqrt{2}$. 4. $\frac{1}{2}\sqrt{7}$. 5. $2\sqrt{3}$. 6. $\frac{1}{2}\sqrt{2}$.
7. $\frac{3}{2}\sqrt[3]{2}$. 8. $\frac{5}{3xy}\sqrt{3xy}$. 9. $\frac{1}{2a}\sqrt[4]{2a^2}$. 10. $\frac{5}{12}\sqrt{6}$. 11. $\frac{x}{y}\sqrt[3]{y}$. 12. $\frac{1}{b}\sqrt{ab}$.

Page 221. — 13. $2\sqrt{7}$. 14. $3\sqrt{2}$. 15. 2. 16. $2\sqrt{3}$. 17. $\frac{1}{2}\sqrt{3}$.
18. $\frac{3\sqrt{2}-2\sqrt{3}}{6}$. 19. $\frac{1}{10}\sqrt{30}$. 20. $\frac{5+\sqrt{7}}{3}$. 21. $\sqrt{10}+3$. 22. $3\sqrt{10}$
$-3\sqrt{3}$. 23. $\sqrt{10}-\sqrt{6}$. 24. $\frac{7+3\sqrt{5}}{2}$. 25. $5+2\sqrt{6}$. 26. $\frac{a+2\sqrt{ab}+b}{a-b}$.
27. $\frac{30+13\sqrt{6}}{19}$. 28. $\frac{5\sqrt{70}-2\sqrt{21}}{238}$. 29. $\frac{6-\sqrt{6}}{5}$. 30. $\frac{2-\sqrt{2}+\sqrt{6}}{4}$.
31. $\frac{2+\sqrt{2}+\sqrt{6}}{4}$. 32. $\frac{12+9\sqrt{3}+3\sqrt{5}-6\sqrt{15}}{22}$.

Page 222. — **35.** .707+. **36.** 5.7735+. **37.** 1.4433+. **38.** .9622+. **39.** −5.121+. **40.** .464+. **41.** .169+. **42.** .127+. **43.** 1.5892+. **44.** 1.632+. **45.** 6.854+. **46.** 2. **3.** 16. **4.** $25\sqrt{3}$. **5.** $2\sqrt{3}$. **6.** $2\sqrt[3]{4}$. **7.** $4\sqrt[4]{2}$. **8.** $x^2\sqrt[3]{x^2}$. **9.** a^3b^9. **10.** $16\,a^5b^2\sqrt[3]{ab^2}$. **11.** a^4x^4. **12.** $16\,a^6b^2x^6$. **13.** $256(a-b)\sqrt[3]{a-b}$. **14.** $x^n\sqrt[m]{y}$.

Page 223. — **16.** 2. **17.** $\sqrt{3}$. **18.** $5\sqrt{2}$. **19.** $5\sqrt[6]{6}$. **20.** $\sqrt{a}$. **21.** $\sqrt{x}$. **22.** $\sqrt[12]{a^3bx}$. **23.** $a^{-1}\sqrt[4]{a^{-1}}$. **24.** $\sqrt[3n]{a^3x}$.

Page 224. — **3.** $\sqrt{5}+1$. **4.** $\frac{1}{2}\sqrt{38}+\frac{1}{2}\sqrt{2}$. **5.** $2+\sqrt{3}$. **6.** $\frac{1}{2}\sqrt{42}-\frac{1}{2}\sqrt{2}$. **7.** $2\sqrt{5}+1$. **8.** $\sqrt{5}+\sqrt{2}$. **9.** $3-\sqrt{2}$. **10.** $\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{12}}$, or $\frac{1}{2}\sqrt{2}+\frac{1}{6}\sqrt{3}$. **11.** $\frac{1}{5}\sqrt{35}-\frac{1}{5}\sqrt{5}$. **12.** $\frac{1}{2}\sqrt{3}+\frac{1}{4}\sqrt{2}$. **13.** $\sqrt{5\,a}+\sqrt{a}$. **14.** $\dfrac{3}{\sqrt{3}+\sqrt{2}}$, or $3(\sqrt{3}-\sqrt{2})$. **15.** $\dfrac{11}{\frac{1}{2}\sqrt{34}-\frac{1}{2}\sqrt{2}}$, or $\frac{11}{16}(\sqrt{34}+\sqrt{2})$.

Page 225. — **17.** $\sqrt{5}+\sqrt{2}$. **18.** $1+\sqrt{5}$. **19.** $\sqrt{5}+\sqrt{6}$. **20.** $3-\sqrt{3}$. **21.** $\sqrt{5}+\sqrt{11}$. **22.** $\sqrt{3}-\sqrt{2}$.

Page 226. — **4.** $x=32$. **5.** $x=9$. **6.** $x=5$. **7.** $x=25$. **8.** $x=2$. **9.** $x=16\frac{7}{8}$. **10.** $x=6$. **11.** $x=12$. **12.** $x=25$. **13.** $x=4$. **14.** $x=5$. **15.** $x=4$. **16.** $x=27$.

Page 227. — **17.** $x=6\,a$. **18.** $x=\dfrac{b^2-a^2}{2\,b}$. **19.** $x=0$. **20.** $x=2$. **21.** $x=a^2-2\,ab+b^2$. **22.** $x=\dfrac{25\,a}{16}$. **23.** $x=\dfrac{a^4+b^2}{2\,b}$. **24.** $x=\dfrac{5\,a}{4}$. **25.** $x=2$. **26.** $x=a$. **27.** $x=\sqrt{a^2-1}$. **28.** $x=\dfrac{a(b-1)^2}{4\,b}$. **29.** $x=\dfrac{a^4-b^4}{2\,a^2}$. **30.** $x=10$. **31.** $x=4$. **1.** $(\frac{3}{4}xy-\frac{1}{3}ax^3)\sqrt{xy}$. **2.** $\dfrac{4}{x+y}\sqrt{x^2-y^2}$. **3.** $(x+2\,a)\sqrt{a}$. **4.** $\frac{1}{6}\sqrt{6}-\frac{1}{3}\sqrt[6]{24}$. **5.** $\dfrac{a}{6}\sqrt[6]{108}$. **6.** $\sqrt{a^2-b}$. **7.** $\dfrac{x^2+xy-x\sqrt{y}-y\sqrt{y}}{x^2-y}$. **8.** $\dfrac{(bx-ax)\sqrt{ab}-ab(x^2-1)}{b^2x^2-ab}$. **9.** $x+\sqrt{x^2-1}$. **10.** $\dfrac{\sqrt{x^2-y^2}-x}{y}$. **11.** $\dfrac{x+21-10\sqrt{x-4}}{x-29}$. **12.** $\sqrt{x^2-y^2}$. **13.** $\sqrt{3}+2\sqrt{2}$. **14.** $\frac{5}{2}\sqrt{2}+\frac{1}{2}\sqrt{14}$. **15.** $\frac{3}{2}\sqrt{6}+\frac{1}{2}\sqrt{30}$. **16.** $2\sqrt{5}-1$. **17.** $\dfrac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{4}$. **18.** $\dfrac{\sqrt{6}+\sqrt{2}-2}{2}$. **19.** $\dfrac{2\sqrt{5}}{3}$. **20.** $\dfrac{\sqrt{7}-3}{6}$.

Page 228. — **21.** $12\sqrt[12]{16200}$. **22.** $-2\,xy-y^2$. **23.** $a^4-a-2\sqrt{ab}-b$. **24.** $\dfrac{1}{a-b}\sqrt{a^2-b^2}$. **25.** $x=\pm\frac{1}{2}\sqrt{3}$. **26.** $x=\dfrac{4\,a^2c^2}{4\,b^2+c^4}$. **27.** $x=\dfrac{2\,ac}{c^2+1}$. **28.** $x=a-1$. **29.** $x=a+b$. **30.** $x=9$.

Page 229. — **1.** $2\sqrt{-1}$. **2.** $5\sqrt{-1}$. **3.** $16\sqrt{-1}$. **4.** $4\,a\sqrt{-1}$. **5.** $x^4\sqrt{-1}$. **6.** $\dfrac{x}{2}\sqrt{-1}$. **7.** $\dfrac{3x}{4}\sqrt{-1}$. **8.** $\dfrac{x^2}{5}\sqrt{-1}$. **9.** $a^2x\sqrt{-1}$. **10.** $(a+b)\sqrt{-1}$.

Page 230. — **3.** $16\sqrt{-1}$. **4.** $2\sqrt{-1}$. **5.** $\frac{1}{10}\sqrt{-1}$. **6.** $10\sqrt{-1}$. **7.** $7\sqrt{-1}$. **8.** $\frac{5}{8}\sqrt{-1}$. **9.** $2x\sqrt{-1}$. **10.** $\frac{1}{a}\sqrt{-1}$.

Page 231. — **3.** $-\sqrt{15}$. **4.** $-15\sqrt{6}$. **5.** -432. **6.** $-512\sqrt{3}$. **7.** $-\frac{1}{2}$. **8.** $15\sqrt{ab}$. **9.** $-6\sqrt{-3}$. **10.** $-6\sqrt{-1}$. **11.** $\sqrt{-2}$. **12.** $\frac{1}{5}\sqrt{10}$. **13.** $-\sqrt{-3}$. **14.** $-\frac{2}{3}$. **15.** $-\sqrt{-b}$. **16.** $-\frac{3}{4}$.

Page 233. — **17.** 17. **18.** 7. **19.** $2a^2+b+b\sqrt{-1}$. **20.** $b(c^2-a^2)$. **21.** $-\sqrt{6}-2+2\sqrt{3}+2\sqrt{2}$. **22.** $4\frac{1}{36}$. **23.** $x^2+\frac{3}{4}$. **24.** b^3-a^3. **25.** $5+2\sqrt{6}$. **26.** $\frac{13-20\sqrt{-3}}{37}$. **27.** $\frac{12\sqrt{-2}-1}{17}$. **28.** $\frac{a^2+ax\sqrt{-1}}{a^2+x^2}$. **29.** $5+2\sqrt{6}$. **30.** $\frac{a+2\sqrt{ab}+b}{a-b}$. **31.** $\frac{5+10\sqrt{-2}}{33}$. **32.** $\frac{4ax\sqrt{-1}}{a^2+x^2}$. **33.** $\frac{(6x^2-2)\sqrt{-1}}{x^2+1}$. **34.** $2\sqrt{-1}$. **35.** $2+3\sqrt{-3}$. **36.** $7-6\sqrt{-2}$. **37.** $-7+4\sqrt{3}$. **38.** $-30+10\sqrt{5}$. **39.** $4a\sqrt{-b}$. **40.** $-x^3-2xy\sqrt{xy}-y^3$. **41.** -1.

Page 236. — **1.** a^4b^4c. **2.** $3a^4b$. **3.** $a^{-1}b^{-1}c$. **4.** $x^{\frac{1}{2}}y^2z^{\frac{1}{3}}$. **5.** $8a^4b$. **6.** $a^2x^{-\frac{1}{2}}y^2$. **7.** $2a^{\frac{1}{n}}b^nc$. **8.** $x^{\frac{2}{n}}y^nz^3$. **9.** abx^2. **10.** $ab^2x^2y^3$. **11.** $\frac{xy^2}{a^2b}$. **12.** $\frac{b^2}{a^2x^3y^5}$. **13.** $ab^2c^2x^3y^5$. **14.** $b^4x^{\frac{1}{3}}$. **15.** $\frac{n^2z^2}{m^4x^2}$. **16.** $a^{\frac{2}{3}}b^{\frac{1}{3}}$. **17.** $a^{\frac{1}{2}}x^{\frac{3}{4}}y^{\frac{1}{4}}$. **18.** $(-27)^{\frac{1}{3}}$. **19.** $16^{\frac{1}{3}}a^{\frac{2}{3}}x^{\frac{3}{3}}$. **20.** $a^{\frac{2}{3}}\times a^{\frac{3}{2}}$. **21.** $a^{\frac{3}{2}}\div a^{\frac{2}{3}}$. **22.** $a^{\frac{2}{3}}b^{-\frac{1}{6}}c^{-\frac{2}{3}}$. **23.** $a^{\frac{2}{5}}b^{\frac{5}{3}}+a^{\frac{2}{3}}b^{\frac{3}{2}}$. **24.** $(a^4b^2\div a^{-4}b^{-2})^{\frac{1}{3}}$. **25.** $\sqrt[3]{a^2b}$. **26.** $\frac{1}{b^2\sqrt[3]{a}}$. **27.** $\frac{\sqrt{5}}{\sqrt{x}}$. **28.** $\frac{2b^2}{\sqrt{a}}$. **29.** $\frac{a^2}{\sqrt[3]{a}}$. **30.** $\frac{\sqrt{4}}{9}$. **31.** $\sqrt[n]{a}$. **32.** $\frac{\sqrt[3]{a}}{c^2\sqrt{b}}$. **33.** 4. **34.** 64. **35.** 9. **36.** $\frac{1}{8}$. **37.** $\frac{1}{64}$. **38.** 3. **39.** 4. **40.** $\frac{1}{100}$. **41.** 100. **42.** $\frac{1}{3}$. **43.** $13\frac{5}{9}$. **44.** 8.

Page 238. — **5.** $a^{\frac{4}{3}}-b^{\frac{4}{3}}$. **6.** $a^{\frac{5}{3}}-b^{-\frac{4}{3}}$. **7.** x^2+2+x^{-2}. **8.** $2+x^{-\frac{1}{2}}y^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}$. **9.** $x-y$. **10.** $x^{-1}-1$. **11.** $x^2+2x^{\frac{3}{2}}+x-4$. **12.** $1+x^{-4}+x^4$. **13.** $a^{\frac{1}{2}}-b^{\frac{1}{2}}$. **14.** $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. **15.** $9x^{-\frac{4}{3}}-6x^{-\frac{2}{3}}y^{-1}+4y^{-2}$. **16.** $x-x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{2}{3}}-y$. **17.** $a+b$. **18.** $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$. **19.** $2a^{\frac{1}{2}}-b^{\frac{1}{2}}$. **20.** $x^{-\frac{1}{2}}+y^{-\frac{1}{3}}$. **21.** $x^{\frac{5}{6}}-2x^{\frac{1}{2}}-x^{\frac{1}{6}}$. **22.** $x^{-2}-\frac{1}{2}x^{-1}y^{-1}+y^{-2}$. **23.** a^{-8} or $a^{\frac{1}{8}}$. **24.** $3^{\frac{n}{2}}-\frac{1}{3^{\frac{n}{2}}}$. **25.** $y^{\frac{1}{2}}$. **26.** $a^{-2}b^{-\frac{1}{2}^3}c^{-12}$ or $\frac{1}{a^2b^{\frac{1}{2}^3}c^{12}}$. **27.** a^ax^{a+b}. **28.** $\frac{1}{b^{-m}}$ or b^m. $3^{\frac{n}{2}}$

Page 241. — **4.** $x=\pm 9$. **5.** $x=\pm 2$. **6.** $x=\pm 3$. **7.** $x=\pm 4$. **8.** $x=\pm 3$. **9.** $x=\pm 5$. **10.** $x=\pm 7$. **11.** $x=\pm 1$. **12.** $x=\pm\sqrt{-5}$. **13.** $x=\pm 7$. **14.** $x=\pm 2\sqrt{5}$. **15.** $x=\pm\sqrt[4]{68}$. **16.** $x=\pm 3\sqrt{-1}$. **17.** $x=\pm 3$. **18.** $x=\pm\sqrt{11}$. **19.** $x=\pm 2\sqrt{-1}$. **20.** $x=\pm\sqrt{2}$. **21.** $x=\pm\frac{5}{13}\sqrt{161}$.

Page 242.—**22.** $x = \pm \frac{4}{3}$. **23.** $x = \pm \sqrt{m+n}$. **24.** $x = 0$. **25.** $x = \pm \frac{a}{1-a^2}\sqrt{1-a^2}$. **26.** $x = \pm \frac{10}{7}\sqrt{7}$. **27.** $x = \pm \frac{a}{b-2c}\sqrt{b^2-4c^2}$. **28.** $x = \pm \frac{1}{c}\sqrt{b^2c^2-2abc}$. **29.** $x = \pm \frac{1}{2}$. **30.** $x = \pm 5$. **31.** 12 and 20. **32.** 6 and 21. **33.** Width, 26.7+ rods; length, 35.6+ rods. **34.** Width, 24 rods; length, 60 rods. **35.** 10 and 6. **36.** Father's age, 40 yrs.; son's, 16 yrs. **37.** $\frac{c}{b}\sqrt{ab}$ and $\frac{c}{a}\sqrt{ab}$. **38.** Son's age, $\frac{1}{b}\sqrt{abm}$; father's, $\frac{1}{a}\sqrt{abm}$.

Page 243.—**2.** 7 and 2. **3.** -1 and 5. **4.** 6 and 4. **5.** $\frac{1}{2}$ and -2. **6.** $-\frac{1}{3}$ and $\frac{1}{2}$. **7.** b and $-c$. **8.** a and $\sqrt{b}$. **9.** $-\sqrt{a}$ and $\sqrt{a}$.

Page 244.—**12.** $x = 4$ or -3. **13.** $x = 9$ or -5. **14.** $x = 7$ or 3. **15.** $x = 8$ or 4. **16.** $x = -7$ or 4. **17.** $x = -8$ or 7. **18.** $x = 14$ or -4. **19.** $x = -13$ or -2. **20.** $x = 22$ or -7. **21.** $x = 2$ or -5. **22.** $x = -\frac{8}{3}$ or 5. **23.** $x = \frac{3}{5}$ or -6.

Page 245.—**24.** $x = -\frac{4}{3}$ or $\frac{5}{2}$. **25.** $x = \frac{7}{4}$ or 1. **26.** $x = \frac{3}{4}$ or 5. **27.** $x = -a$ or $-b$. **28.** $x = a$ or $2a$. **29.** $x = a$ or $-\frac{b}{2}$. **30.** $x = -\sqrt{a}$ or $-\sqrt{b}$. **31.** -3. **32.** 2 and 3. **33.** -5 and 2. **34.** 12 and -5. **35.** -1 and -4. **36.** -1 and 5.

Page 246.—**39.** $x^2 + 4x - 5 = 0$. **40.** $x^2 + 7x + 12 = 0$. **41.** $2x^2 + 3x - 2 = 0$. **42.** $12x^2 - x - 1 = 0$. **43.** $6x^2 + 7x + 2 = 0$. **44.** $x^2 + 4x = 0$. **45.** $x^2 - 5 = 0$. **46.** $x^2 - 3 = 0$. **47.** $x^2 - 4x + 1 = 0$. **48.** $x^2 + \left(a - \frac{a}{b}\right)x - \frac{a^2}{b} = 0$.

Page 248.—**4.** $x = 5$ or 3. **5.** $x = -2$ or -10. **6.** $x = 3$ or -7. **7.** $x = 15$ or -3. **8.** $x = 19$ or 1. **9.** $x = 4$ or -22. **10.** $x = 4$ or 3. **11.** $x = 5$ or $-\frac{4}{3}$. **12.** $x = 4 \pm \sqrt{-1}$. **13.** $x = 10$ or $-8\frac{2}{7}$. **14.** $x = -2$ or -3. **15.** $x = -5$ or -42. **16.** $x = 4$ or 3. **17.** $x = 7$. **18.** $x = 7$ or $-\frac{11}{37}$. **19.** $x = 15 \pm 4\sqrt{14}$. **20.** $x = \frac{9}{14} \pm \frac{1}{14}\sqrt{67}$. **21.** $x = \pm 5$. **22.** $x = 5$ or -2. **23.** $x = \pm 5$.

Page 249.—**2.** $x = 4$ or $-\frac{5}{2}$. **3.** $x = 2$ or $\frac{2}{3}$. **4.** $x = -1$ or $-\frac{1}{3}$. **5.** $x = 3$ or 1.

Page 250.—**6.** $x = -\frac{1}{3}$ or $-\frac{2}{3}$. **7.** $x = 2$ or $-\frac{8}{9}$. **8.** $x = 9$ or $-\frac{1}{9}$. **9.** $x = 7$ or -5. **10.** $x = 4$ or $-\frac{17}{8}$. **11.** $x = 3$ or $\frac{7}{6}$. **12.** $x = \frac{9}{10}$ or $-\frac{3}{5}$. **13.** $x = \frac{13}{6}$ or $-\frac{2}{3}$.

Page 251.—**15.** $x = -\frac{4}{3}$ or $\frac{2}{3}$. **16.** $x = 5$ or $-4\frac{1}{3}$. **17.** $x = 4\frac{3}{5}$ or -3. **18.** $x = \frac{3}{2}$ or $\frac{4}{5}$. **19.** $x = -\frac{1}{7}$ or $-\frac{1}{3}$. **20.** $x = \frac{1}{3}$ or $\frac{1}{5}$. **21.** $x = 4$ or $-\frac{13}{10}$. **22.** $x = 2$ or $-\frac{12}{11}$. **23.** $x = 4$ or $-4\frac{2}{3}$. **24.** $x = 2$ or $-\frac{4}{5}$.

Page 252.—**3.** $x = 37$ or -13. **4.** $x = 48.2173$ or -7.2173. **5.** $x = 17$ or 7. **6.** $x = 4$ or $-\frac{1}{3}$.

Page 253.—**7.** $x = 7$ or -26. **8.** $x = 6$ or $-\frac{7}{4}$. **9.** $x = 2$ or $-6\frac{1}{5}$. **10.** $x = -\frac{6}{7}$ or -4. **11.** $x = 5$ or $-\frac{4}{3}$. **12.** $x = 1 \pm \frac{1}{2}\sqrt{-6}$. **13.** $x = \frac{-n \pm \sqrt{n^2+16m}}{2m}$. **14.** $x = \frac{-1 \pm \sqrt{5}}{2m}$.

Page 254. — **4.** $x = a$ or 1. **5.** $x = \frac{a}{b}$ or $\frac{b}{a}$. **6.** $x = a^2$ or b^2. **7.** $x = 2$ or $-\frac{1}{a}$. **8.** $\frac{a}{2}(1 \pm \sqrt{5})$. **9.** $x = n$ or 1. **10.** $x = a$ or $-\frac{a}{a+1}$. **11.** $x = a$ or b. **12.** $x = 2ab \pm \sqrt{a^2 - b^2 + 4a^2b^2}$. **13.** $x = 0$ or $-(a+2)$. **14.** $x = a \pm \frac{1}{a}$. **15.** $x = 2a$ or $-b$. **16.** $x = a$ or $-(b+c)$. **17.** $x = c + d \pm \sqrt{6cd - 2c^2}$. **18.** $x = \frac{2a}{a+b}$ or $-\frac{2b}{a+b}$. **19.** $x = m$ or $\frac{1}{m}$.

Page 255. — **1.** $x = 17$ or 4. **2.** $x = 14$ or -13. **3.** $x = \frac{5}{2}$ or -1. **4.** $x = 2$ or $\frac{2}{3}$. **5.** $x = \frac{2}{3}$. **6.** $x = 5$ or $-6\frac{2}{5}$. **7.** $x = 1\frac{1}{2} \pm 1\frac{1}{2}\sqrt{13}$. **8.** $x = a$ or $2a$. **9.** $x = \frac{1}{2}$ or -2. **10.** $x = \frac{1}{3}$ or $-\frac{1}{4}$. **11.** $x = -\frac{1}{2}$ or $-\frac{2}{3}$. **12.** $x = 0$ or -4. **13.** $x = \frac{1}{2}$ or $-\frac{2}{3}$. **14.** $x = \frac{5}{2}$ or -2. **15.** $x = a$ or $-\frac{1}{a}$. **16.** $x = \pm\frac{a}{2}\sqrt{2}$. **17.** $x = \frac{a}{b}$ or a. **18.** $x = a \pm \sqrt{b}$. **19.** $x = a + b$ or $\frac{a^2 + b^2}{a+b}$. **20.** $x = b$ or $\frac{a^2}{b}$. **21.** $x = b$ or $-a$. **22.** $x = c$ or $\frac{2c}{a}$. **23.** $x = 2a - b$ or $-b$. **24.** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{4a^2 + 12a + 1}$. **25.** $x = 2 \pm \sqrt{3}$. **26.** $x = 1 \pm \sqrt{-2}$. **27.** $x = 2a$ or -12. **28.** $x = \frac{3}{5} \pm \frac{1}{5}\sqrt{9 - 8a^2}$. **29.** $x = \frac{3}{2}$ or $\frac{2}{3}$. **30.** $x = \frac{3}{2}$ or $-\frac{9}{10}$. **31.** $x = 36$ or 12. **32.** $x = 1\frac{4}{5}$ or 2.

Page 256. — **33.** $x = 5$ or 21. **34.** $x = -4\frac{3}{8} \pm \frac{5}{8}\sqrt{241}$. **35.** $x = 4$ or -1. **36.** $x = 4$ or $\frac{7}{3}$. **37.** $x = \frac{22 \pm 2\sqrt{-319}}{5}$. **38.** $x = \frac{2n \pm \sqrt{4n^2 - 3}}{3}$. **39.** $x = a$ or -1. **40.** $x = 13a$ or $-6a$. **41.** $x = a$ or $-2a$. **42.** $x = \frac{n}{2} \pm \frac{1}{2}\sqrt{n^2 + 4}$. **43.** $x = \frac{c}{2a} \pm \frac{1}{2a}\sqrt{c^2 - 4ab}$. **44.** $x = \frac{m+n}{m-n}$ or $\frac{n-m}{n+m}$. **45.** $x = m \pm n$. **46.** $x = \frac{a \pm \sqrt{b}}{2c}$. **47.** $x = 9$ or $\frac{1}{4}$. **48.** $x = 1$ or $-\frac{1}{2}$. **49.** $x = 5 \pm 2\sqrt{6}$. **50.** $x = \frac{ab}{a+b}$. **51.** $x = \frac{9}{4}$ or $\frac{1}{4}$. **52.** $x = 9$ or 5. **53.** $x = 0$. **54.** $x = \frac{4}{5}$. **55.** $x = c$ or $\frac{1}{c}$. **56.** $x = -\frac{a}{2} \pm \frac{a}{2}\sqrt{4a^2 + 9}$. **57.** $x = 10$. **58.** $x = 4$. **59.** $x = 6$ or $-\frac{10}{9}$. **60.** $x = 4$ or 0. **61.** $x = 2$ or $-\frac{4}{15}$. **62.** $x = \frac{1}{2} \pm \frac{1}{14}\sqrt{3689}$. **63.** $x = \frac{a+b}{a-b}$ or $\frac{a-b}{a+b}$. **64.** $x = 8$ or $-\frac{23}{5}$. **65.** $x = \pm\sqrt{3}$. **66.** $x = \frac{1}{2}(1 \pm \sqrt{5})$.

Page 259. — **6.** $x = \pm 3$ or ± 1. **7.** $x = \pm 5$ or $\pm\frac{1}{3}\sqrt{3}$. **8.** $x = \pm 2\sqrt{2}$ or $\pm\sqrt{-2}$. **9.** $x = 2$ or $\sqrt[3]{-6}$. **10.** $x = 2$ or $\sqrt[3]{-4}$. **11.** $x = \sqrt[2n]{p} \pm \sqrt{q + p^2}$. **12.** $x = \pm\sqrt{3}$; ± 2 or $\pm\sqrt{-1}$. **13.** $x = \pm 1$ or $\pm\sqrt[4]{\frac{5}{7}}$. **14.** $x = 18$ or 3. **15.** $x = 25$ or 3. **16.** $x = \pm\sqrt{2}$. **17.** $x = \frac{9 \pm 2\sqrt{19}}{5}$. **18.** $x = 1\frac{6}{49}$ or 0. **19.** $x = 4$ or $\sqrt[3]{49}$. **20.** $x = 4$ or -1. **21.** $x = 3$ or $\sqrt[3]{-7}$. **22.** $x = 9$ or -2; $\frac{1}{2}(7 \pm \sqrt{173})$. **23.** $x = 1 \pm 2\sqrt{15}$ or 1.

24. $x=\frac{1}{8}(9\pm\sqrt{-35})$ or $\frac{1}{2}(3\pm\sqrt{7})$. **25.** $x=13$ or 78. **26.** $x=\frac{24}{25}$ or 0. **27.** $x=\sqrt[n]{2a^2\pm 2a\sqrt{a^2+b^2}+b^2}$. **28.** $x=\pm 8$ or $\sqrt{(-\frac{1}{6}^9)^3}$. **29.** $x=\pm 1$. **30.** $x=a$ or $\frac{1}{5}a$.

Page 260. — **31.** $x=5$, 2, and 3. **32.** $x=2$, 2, and -3. **33.** $x=-1$, 1, 2, and -2; or $x=\pm 1$ and ± 2. **34.** $x=2$, -3, 1, and 4.

Page 262. — **6.** 12 and 5. **7.** 60 and 12. **8.** 29 and 21. **9.** 45 and 52, or 52 and -45. **10.** $\frac{m}{2}\pm\frac{1}{2}\sqrt{m^2-4n}$; $\frac{m}{2}\mp\frac{1}{2}\sqrt{m^2-4n}$.

11. $\frac{m}{2}\pm\frac{1}{2n}\sqrt{m^2n^2-4mn}$; $\frac{m}{2}\mp\frac{1}{2n}\sqrt{m^2n^2-4mn}$. **12.** 9 or -11. **13.** 18, 9, and 6.

Page 263. — **14.** Persons, 19; amount each paid, \$10. **15.** Breadth, 34 rods; length, 35 rods. **16.** First square, 5625 men; second square, 7225 men. **17.** 4 miles per hour. **18.** 16 sheep. **19.** Age, 36 yrs. **20.** A, 10 miles per hour; B, 9 miles per hour. **21.** 25 robes. **22.** $118\frac{3}{4}$ sq. rds. **23.** 63 acres; selling price, \$50 per acre.

Page 264. — **24.** 15 pieces; cost of each, \$84. **25.** 12 persons. **26.** \$80. **27.** 12 horses. **28.** One pipe, 15 hours; other pipe, 10 hours. **29.** 15 miles per hour. **30.** 600 lbs. **31.** 84 sheep. **32.** 20 cts. per dozen; $x=\frac{-ac\pm\sqrt{a^2c^2+48abc}}{2a}$.

Page 266. — **2.** $x=8$ or -5; $y=5$ or -8. **3.** $x=5$ or $12\frac{1}{3}$; $y=3$ or $-11\frac{2}{3}$. **4.** $x=8$ or $\frac{4}{3}$; $y=3$ or $-\frac{1}{3}$. **5.** $x=5$ or $13\frac{2}{3}$; $y=4$ or $-13\frac{1}{3}$. **6.** $x=5$ or $-4\frac{2}{3}\frac{5}{1}$; $y=4$ or $-3\frac{1}{3}\frac{1}{1}$. **7.** $x=1$ or $\frac{7}{4}$; $y=2$ or $\frac{4}{7}$. **8.** $x=4$ or $1\frac{2}{7}\frac{9}{7}$; $y=5$ or $8\frac{2}{4}\frac{7}{7}$. **9.** $x=2$ or $-\frac{1}{3}$; $y=4$ or $\frac{5}{3}$. **10.** $x=1$ or $2\frac{8}{17}$; $y=3$ or $1\frac{4}{17}$. **11.** $x=3$ or $-1\frac{8}{15}$; $y=2$ or $-9\frac{1}{3}$. **12.** $x=5$ or $-1\frac{2}{61}$; $y=6$ or $-14\frac{3}{61}$.

Page 267. — **14.** $x=\pm 3$ or $\pm\frac{6}{7}\sqrt{7}$; $y=\pm 2$ or $\pm\frac{1}{7}\sqrt{7}$. **15.** $x=\pm 1$ or $\mp\sqrt{2}$; $y=\pm 2$ or $\pm\frac{1}{2}\sqrt{2}$. **16.** $x=\pm 3$ or $\mp\frac{3}{7}\sqrt{7}$; $y=\pm 1$ or $\pm\frac{6}{7}\sqrt{7}$. **17.** $x=\pm 4$ or $\mp\frac{8}{7}\sqrt{7}$; $y=\pm 2$ or $\pm\frac{2}{7}\sqrt{7}$. **18.** $x=\pm 6$ or $\mp\frac{1}{2}\sqrt{2}$; $y=\pm 5$ or $\pm 1\frac{1}{2}\sqrt{2}$. **19.** $x=\pm 3$ or $\mp\frac{2}{3}\sqrt{-21}$; $y=\pm 2$ or $\pm\frac{3}{7}\sqrt{-21}$. **20.** $x=\pm 4$ or $\pm 2\sqrt{3}$; $y=\pm 3$ or $\pm\sqrt{3}$. **21.** $x=\pm 3$ or $\pm\frac{5}{2}\sqrt{2}$; $y=\pm 2$ or $\pm\frac{1}{2}\sqrt{2}$. **22.** $x=\pm 3$ or $\mp\frac{1}{6}\sqrt{2}$; $y=\pm 2$ or $\pm\frac{7}{6}\sqrt{2}$. **23.** $x=\pm 8\sqrt{\frac{1}{6}\frac{5}{3}\frac{8}{3}}$ or $\pm\frac{9}{28}\sqrt{3871}$; $y=\pm\sqrt{\frac{1}{6}\frac{5}{3}\frac{8}{3}}$ or $\pm\frac{1}{4}\sqrt{3871}$.

Page 269. — **27.** $x=7$ or -5; $y=5$ or -7. **28.** $x=5$ or -3; $y=-3$ or 5. **29.** $x=5$ or -6; $y=-6$ or 5. **30.** $x=3$ or 2; $y=2$ or 3. **31.** $x=4$ or -3; $y=\frac{3}{2}$ or -2. **32.** $x=5$ or -2; $y=2$ or -5. **33.** $x=4$ or 3; $y=3$ or 4. **34.** $x=8$; $y=5$. **35.** $x=5$ or -2; $y=2$ or -5. **36.** $x=4$ or -3; $y=3$ or -4. **37.** $x=6$ or -5; $y=-5$ or 6. **38.** $x=10$ or -5; $y=5$ or -10.

Page 270. — **4.** $x=5$; $y=3$. **5.** $x=9$; $y=7$. **6.** $x=3$ or -2; $y=2$ or -3. **7.** $x=7$ or 5; $y=5$ or 7. **8.** $x=\pm 5$; $y=\pm 3$. **9.** $x=\pm 6$; $y=\pm 2$.

Page 271. — **10.** $x=5$ or 3; $y=3$ or 5. **11.** $x=5$ or -2; $y=-2$ or 5. **12.** $x=2$ or 1; $y=1$ or 2. **13.** $x=3$ or 2; $y=2$ or 3; $x=$

$-3\pm$
4; y -
$y=-$
and 9.

Pag
13. I.
125 f
pipe,

Pa

Pa

8. x

P
3. f
9; f
9.

I
6.

4.

5.

$
]
a

Page 298. — **1.** 1020. **2.** -336. **3.** 369. **4.** 35. **5.** 237.5. **6.** 5050. **7.** $\frac{n+n^2}{2}$.

Page 299. — **8.** n^2. **9.** (in 7) n; (in 8) $2n-1$. **10.** $s=240$. **11.** $d=-\frac{1}{4}$. **12.** $s=143$. **13.** $l=18\frac{1}{2}$. **14.** $a=45$. **15.** $n=11$. **16.** $a=-3$; $s=0$. **17.** \$440, \$480, \$520, and \$560. **18.** 12 yards. **19.** 140 days. **20.** 29700 yards.

Page 301. — **5.** 128. **6.** $\frac{1}{32768}$.

Page 302. — **7.** $\frac{1}{15625}$. **8.** $a=6$. **9.** $a=15$. **10.** $r=3$. **11.** $r=\frac{2}{3}$. **12.** $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}$. **13.** $\frac{1}{9}, \frac{\sqrt{3}}{9}, \frac{1}{3}, \frac{\sqrt{3}}{3}, 1, \sqrt{3}, 3, 3\sqrt{3}, 9$. **14.** $\frac{1}{5}, \frac{2\sqrt{7}}{35}, \frac{28}{245}, \frac{8\sqrt{7}}{245}, \frac{112}{1715}, \frac{32\sqrt{7}}{1715}, \frac{64}{1715}$. **15.** $\frac{a}{x}=\frac{x}{b}$. $\therefore x=\sqrt{ab}$.

Page 304. — **20.** $s=-29524$. **21.** $s=93\frac{405}{625}$. **22.** $s=\frac{1261}{256}$. **23.** $a=3$. **24.** $a=1$. **25.** $a=8$. **26.** $a=979.2$. **27.** $\frac{3}{2}$. **28.** 2. **29.** $1\frac{1}{5}$. **30.** $3\frac{3}{4}$. **31.** $\frac{n}{n-1}$.

Page 305. — **33.** $\frac{238}{990}$. **34.** $6\frac{53}{99}$. **35.** $\frac{39}{110}$. **36.** $\frac{450758}{999900}$. **37.** $8\frac{891}{999}$. **38.** $54\frac{107}{333}$.

Page 306. — **1.** $r=3$; $s=4372$. **2.** $s=635$; $r=2$. **3.** $a=\frac{1}{64}$; $s=1365\frac{21}{64}$. **4.** $a=2\frac{1}{2}$; $s=40957\frac{1}{2}$. **5.** $l=1594323$. **6.** $r=2$; $n=22$. **7.** $s=3$. **8.** $s=\frac{x^2}{x^2+1}$. **9.** $s=2\frac{242}{243}$. **10.** $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{81}{2}$. **11.** $r=\frac{1}{10}$, or 10%. **12.** 2, 10, 50. **13.** $-3, -18, -108, -648$, or $2\frac{1}{7}, -12\frac{6}{7}, 77\frac{1}{7}, -462\frac{6}{7}$. **14.** \$10,000. **15.** 15,625,000,000 grains.

Page 307. — **16.** $89\frac{17}{27}$ feet; 90 feet. **17.** 3, 6, 12, 24, or $-24, -12, -6, -3$.

Page 308. — **3.** 3. **4.** $\frac{bc}{2c-b}$. **5.** $\frac{1}{7}^s$ and $\frac{1}{5}^s$. **6.** $\frac{1}{3}, \frac{3}{14}, \frac{3}{19}, \frac{1}{8}, \frac{3}{29}, \frac{3}{34}, \frac{1}{13}$. **7.** $\frac{2}{3}, \frac{3}{4}, \frac{6}{7}, 1, \frac{6}{5}, \frac{3}{2}$. **8.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}$. **9.** $\frac{1}{6}$.

Page 316. — **5.** 4.95424. **6.** 3.92942. **7.** $\bar{3}.80821$. **8.** 4.50651.

Page 317. — **11.** $\bar{3}.65437$. **12.** 0.93725. **13.** $\bar{1}.89709$. **14.** 4.53866.

Page 318. — **15.** 2, 3, 4, 7, 9. **16.** 30, 500, 6000, .008, .09. **17.** 12, 23, 45, 66, 84. **18.** 135.249; 326.496; 597.493; 73.173; .93119. **19.** 4375.05; 59621.9; 673.22; 8.16056; .00053274.

Page 323. — **6.** 182.97+. **7.** 4.42+. **8.** 6.0115+. **9.** 66.394. **10.** .00037658. **11.** 6.0138+. **12.** .19114. **13.** .000050962. **14.** 19770. **15.** 1.7018. **16.** .0036648. **17.** 1.7489+. **18.** 1.011.

Page 324. — **19.** $n=8$. **20.** $n=6$. **21.** $n=5$. **22.** $n=5$. **23.** $x=3$. **24.** $x=3$. **25.** $x=1\frac{1}{2}$, nearly. **26.** $x=1.029$. **27.** $x=1.59+$. **28.** $x=\frac{\log c-\log a}{\log b}$.

Page 325. — **2.** 8%. **3.** \$450. **4.** $n=\frac{\log m}{\log(1+r)}$.

Page 328. — **3.** \$8456.52. **4.** \$4856$\frac{3}{4}$. **5.** \$3322. **6.** 23.98 yrs. **7.** 23.44+ yrs. **8.** \$145985.20. **9.** \$1554.04. **10.** \$106.63. **11.** \$125. **12.** \$27272$\frac{8}{11}$.

Page 332. — **1.** $\frac{1}{x-5}+\frac{1}{x-8}$. **2.** $\frac{31}{x-8}-\frac{29}{x-7}$. **3.** $\frac{3}{x-5}+\frac{4}{x-1}$. **4.** $\frac{7}{x+5}-\frac{3}{x-2}$. **5.** $\frac{2}{3x+1}+\frac{3}{2x+1}$. **6.** $\frac{1}{x+2}+\frac{2}{x+3}-\frac{1}{x}$. **7.** $2-\frac{2}{x+1}+\frac{3}{x-1}$. **8.** $3+\frac{4}{x-1}+\frac{1}{x+1}-\frac{2}{x}$. **9.** $\frac{6}{x-1}+\frac{1}{x+1}-\frac{16}{2x-1}+\frac{2}{2x+1}$. **10.** $\frac{3}{x-3}+\frac{2}{x+3}-\frac{2}{x}$. **11.** $\frac{10}{x+4}-\frac{2}{x+2}$. **12.** $1+\frac{2}{5x-5}-\frac{4}{10x+15}$.

Page 334. — **1.** $1-5x+15x^2-45x^3+135x^4-\cdots$. **2.** $1+2x+3x^2+4x^3+5x^4+\cdots$. **3.** $1+x+5x^2+13x^3+41x^4+121x^5+\cdots$. **4.** $1+2x+x^2-x^3-2x^4-x^5+\cdots$. **5.** $\frac{1}{3}+\frac{1}{9}x+\frac{1}{27}x^2+\frac{1}{81}x^3+\frac{1}{243}x^4+\cdots$. **6.** $1+3x+7x^2+17x^3+41x^4+99x^5+\cdots$. **7.** $1-x-x^2+5x^3-7x^4-x^5+\cdots$. **8.** $a-(2a-b)x+(a-2b)x^2+(4a+b)x^3-(11a-4b)x^4+\cdots$.

Page 338. — **4.** $1-\frac{1}{4}\cdot\frac{1}{16}+\frac{1\cdot5}{4\cdot8\cdot16^2}-\frac{1\cdot5\cdot9}{4\cdot8\cdot12\cdot16^3}+\cdots$. **5.** $1+\frac{1}{3\cdot9}+\frac{1\cdot4}{3\cdot6\cdot9^2}+\frac{1\cdot4\cdot7}{3\cdot6\cdot9\cdot9^3}+\cdots$. **6.** $1+\frac{1}{4\cdot5}-\frac{1\cdot3}{4\cdot8\cdot5^2}+\frac{1\cdot3\cdot7}{4\cdot8\cdot12\cdot5^3}-\cdots$. **7.** $2^{\frac{1}{3}}\left(1+\frac{1}{3\cdot2\cdot4}-\frac{1\cdot2}{3\cdot6\cdot2^2\cdot4^2}+\frac{1\cdot2\cdot5}{3\cdot6\cdot9\cdot2^3\cdot4^3}-\cdots\right)$. **8.** $1+\frac{a}{2}-\frac{a^2}{2\cdot4}+\frac{3a^3}{2\cdot4\cdot6}-\cdots$. **9.** $1-\frac{2a}{3}-\frac{1\cdot2}{3\cdot6}a^2-\frac{1\cdot2\cdot4}{3\cdot6\cdot9}a^3-\cdots$. **10.** $a^{-3}-3a^{-4}b+6a^{-5}b^2-10a^{-6}b^3+\cdots$. **11.** $a^{-\frac{2}{3}}+\frac{2}{3}a^{-\frac{5}{3}}b+\frac{2\cdot5}{3\cdot6}a^{-\frac{8}{3}}b^2+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}a^{-\frac{11}{3}}b^3+\cdots$. **12.** $1+\frac{1}{5}\cdot\frac{1}{5}-\frac{1\cdot4}{5\cdot10\cdot5^2}+\frac{1\cdot4\cdot9}{5\cdot10\cdot15\cdot5^3}-\cdots$.

Page 339. — **2.** 3.036589+. **3.** 2.024398+. **4.** 2.0022248+. **5.** 3.0024648+.

Page 343. — **2.** $x=2$; $y=1$. **3.** $x=5$; $y=2$. **4.** $x=1$; $y=1$. **5.** $x=\frac{c^2-bf}{ac-bd}$; $y=\frac{af-cd}{ac-bd}$. **6.** $10-12=-2$. **7.** 75. **8.** -15.

Page 347. — **9.** $x=3$; $y=2$; $z=1$. **10.** $x=6$; $y=8$; $z=10$. **11.** $x=2$; $y=-2$; $z=0$. **12.** $x=1$; $y=2$; $z=3$.

Page 366. — **4.** 120. **5.** 5040. **6.** 325. **7.** 120 (at round table 24). **8.** 6720. **9.** 13699. **10.** 34650.

Page 368. — **3.** 126. **4.** 84. **5.** 15 (16 incl.−1). **6.** 495. **7.** 57. **8.** 1,392,300. **9.** 35. **10.** 63 in A; 62 in B.

To avoid fine, this book should be returned on
or before the date last stamped below

10M—6.40

TX
512.1
W583

632282

NAME
DATE
NAME
DATE

White, E.E.
School algebra.
W583

LIBRARY, SCHOOL OF EDUCATION, STANFORD

632282

To avoid fine, this book sho
or before the date last